

Candidacy Exam  
 Department of Physics  
 October 1, 2011  
 Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	$N_A$	$6.022 \times 10^{23} \text{ mol}^{-1}$
Boltzmann's constant	$k_B$	$1.381 \times 10^{-23} \text{ J K}^{-1}$
Electron charge magnitude	$e$	$1.602 \times 10^{-19} \text{ C}$
Gas constant	$R$	$8.314 \text{ J mol}^{-1} \text{ K}^{-1}$
Planck's constant	$h$ $\hbar = h/2\pi$	$6.626 \times 10^{-34} \text{ J s}$ $1.055 \times 10^{-34} \text{ J s}$
Speed of light in vacuum	$c$	$2.998 \times 10^8 \text{ m s}^{-1}$
Permittivity constant	$\epsilon_0$	$8.854 \times 10^{-12} \text{ F m}^{-1}$
Permeability constant	$\mu_0$	$1.257 \times 10^{-6} \text{ N A}^{-2}$
Gravitational constant	$G$	$6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^5 \text{ N m}^{-2}$
Stefan-Boltzmann constant	$\sigma$	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Electron rest mass	$m_e$	$9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV } c^{-2}$
Proton rest mass	$m_p$	$1.673 \times 10^{-27} \text{ kg} = 938.3 \text{ MeV } c^{-2}$
Origin of temperature scales		$0^\circ\text{C} = 273 \text{ K}$
1 large calorie (as in nutrition)		4.184 kJ
1 inch		2.54 cm

Definite integrals:

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \quad (\text{I-1})$$

$$\int_0^{\infty} x^n e^{-x} dx = \Gamma(n+1) = n!. \quad (\text{I-2})$$

Laplacian in spherical polar coordinates  $(r, \theta, \phi)$ :

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \quad (\text{I-3})$$

Laplacian in cylindrical coordinates  $(r, \theta, z)$ :

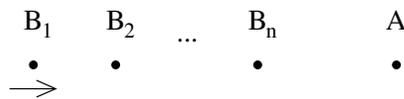
$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \quad (\text{I-4})$$

- I-1. (a) In an experiment, a particle  $A$  of mass  $m$  is at rest on a smooth horizontal table. A particle  $B$  of mass  $bm$ , where  $b > 1$ , is projected along the table directly towards  $A$  with speed  $u$ .



The collision is perfectly elastic. Find an expression for the speed of  $A$  after the collision in terms of  $b$  and  $u$ . What is the maximum possible ratio of the final speed of  $A$  to the initial speed of  $B$ . All motion is along a single line.

- (b) In a second experiment, particles  $B_1, B_2, \dots, B_n$  and  $A$  are in a line on the table as follows:



Particle  $B_1$  is projected directly towards the other particles, with speed  $u$ . The other particles  $B_2, B_3, \dots, B_n$ , and  $A$  are initially at rest. The mass of  $B_i$  ( $i = 1, 2, \dots, n$ ) is  $\lambda^{n+1-i}m$ , where  $\lambda$  is a number that is greater than 1; the mass of  $A$  is  $m$ . All collisions are perfectly elastic. Show that, by choosing  $n$  sufficiently large, there is no upper limit on the speed at which  $A$  can be made to move. In the case  $\lambda = 4$ , determine the least value of  $n$  for which  $A$  moves at more than  $20u$ .

- I-2. (a) In quantum mechanics, the wave function  $\Psi$  for a single particle defines the probability density  $\rho = |\Psi|^2$ . Find an expression for the probability current  $\vec{J}$  such that a continuity equation results.

- (b) Consider a scattering problem in one dimension such that stationary states have the asymptotic form  $\psi(x) = Ae^{ikx} + Be^{-ikx}$  to the left of the scattering potential, and  $\psi(x) = Fe^{ikx} + Ge^{-ikx}$  to the right. The solution of the Schrödinger equation gives a relation between the coefficients of the form

$$\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix}$$

where  $S$  is a matrix called the  $S$ -matrix. Using the results of part (a), show that  $S$  is unitary.

- I-3. (a) Use Maxwell's equations to show that the electromagnetic field satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = 0.$$

- (b) Explain how this equation implies charge conservation.  
(c) What is the analogous equation for energy conservation of the electromagnetic field?

- I-4. A self-contained machine only inputs two equal steady streams of hot and cold water at temperatures  $T_1$  and  $T_2$ . Its only output is a single high-speed jet of water. The heat capacity per unit mass of water,  $C$ , may be assumed to be independent of temperature. The machine is in a steady state, and the kinetic energy in the incoming jets is negligible.

- (a) What is the speed of the outgoing jet in terms of  $T_1$ ,  $T_2$  and  $T$ , where  $T$  is the temperature of the outgoing jet?  
(b) What is the maximum possible speed of the jet?

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Part II

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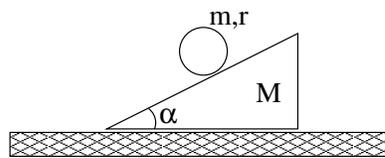
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II-1. A hoop of mass  $m$  and radius  $r$  rolls without slipping down a wedge of mass  $M$ . The wedge makes a fixed angle  $\alpha$  with the horizontal, and it is on a frictionless horizontal surface:



- (a) Derive equations of motion for the system.
  - (b) Obtain an expression for the position and velocity of the wedge as a function of time.
- II-2. (a) An electron of definite momentum starts in a large potential-free region going towards a region where the electrical potential is  $\phi$ . The boundary between the regions is sharp and perpendicular to the momentum of the electron. Find a suitable wave function for this situation.
- (b) An experiment to test this is made by aiming a well-collimated beam of electrons in the potential-free region perpendicularly towards the region boundary. The electrical potential in the other region is  $\phi = -3 \text{ V}$ , the de Broglie wavelength of the electrons is  $0.5 \text{ nm}$ , and the beam current is  $2 \mu\text{A}$ . Determine how much current is reflected at the boundary.
- II-3. A perfectly conducting sphere of radius  $a$  has no net charge and is not grounded. A constant external electric field  $\vec{E} = \hat{z}E$  is applied to the sphere:

- (a) What is the electrostatic potential outside of the sphere?
- (b) What is the surface charge density on the surface of the sphere?
- (c) What is the polarizability of the sphere? (The polarizability  $\alpha$  is the ratio of  $p/E$ , where  $p$  is the net dipole moment of the sphere.)

II-4. Liquid helium-4 has a normal boiling point of  $T_0 = 4.2K$  (at a pressure  $p_0 = 1\text{atm}$ ). However, at a lower pressure  $p_1 = 1.3 \times 10^{-3}\text{atm}$ , it boils at a lower temperature  $T_1 = 1.2K$ .

The Clapeyron-Clausius equation, describing the slope of the phase equilibrium line, is

$$\frac{dp}{dT} = \frac{\Delta s}{\Delta v} \quad (\text{II-5})$$

where  $\Delta s$  and  $\Delta v$  are the differences in molar entropy and molar volume between two phases, respectively. Assuming that gaseous helium is an ideal gas, use this equation to estimate the average latent heat of vaporization for temperatures between  $T_0$  and  $T_1$  in terms of  $p_0$ ,  $p_1$ ,  $T_0$ ,  $T_1$  and the gas constant  $R$ . Evaluate this value for the given pressures and temperatures.