

Qualifying Exam for Ph.D. Candidacy

Department of Physics

February 6, 2021

Part I

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Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV } c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27} \text{ kg} = 938.3 \text{ MeV } c^{-2}$
Origin of temperature scales		$0^\circ \text{C} = 273 \text{ K}$
1 large calorie (as in nutrition)		4.184 kJ
1 GeV		$1.609 \times 10^{-10} \text{ J}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \quad (\text{I-1})$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \quad (\text{I-2})$$

Indefinite integrals:

$$\int \frac{x}{(x^2 + a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2 + a^2)^{n-1}} + c \quad \text{for } n \neq 0, 1 \quad (\text{I-3})$$

Gradient in spherical polar coordinates (r, θ, ϕ) :

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \quad (\text{I-4})$$

Laplacian in spherical polar coordinates (r, θ, ϕ) :

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \quad (\text{I-5})$$

Laplacian in cylindrical coordinates (r, θ, z) :

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Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (\text{I-7})$$

I-1. Two Lagrangians L' and L , which differ by the total time derivative dF/dt of some function $F(q, t)$,

$$L' = L + dF/dt ,$$

are physically equivalent, i.e. they lead to the same Lagrange's equations of motion.

- (a) What is the relation between the generalized momenta p' and p which these two Lagrangians yield?
- (b) What is the relation between the Hamiltonians H' and H which these two Lagrangians yield?
- (c) Show explicitly that Hamiltonian's equations of motion in the primed quantities are equivalent to those in the unprimed quantities.

I-2. In this problem, we consider the optical transition rate of light through graphene. Electrons in graphene are described by two-dimensional massless Dirac fermions with the Hamiltonian $H_0 = \hbar\nu_f(k_x\sigma_x + k_y\sigma_y)$, where σ_x and σ_y are 2×2 Pauli matrices and k_x and k_y are the components of the wave vector.

- a) Find the energies $E_{\pm}(\vec{k})$, of the Hamiltonian H_0 , as well as the corresponding eigenstate wave functions, denoted as ψ_{\pm} .
- b) Consider the coupling of Dirac electrons with electromagnetic fields with the full Hamiltonian

$$H = \hbar\nu_f((k_x + \frac{e}{\hbar}A_x)\sigma_x + (k_y + \frac{e}{\hbar}A_y)\sigma_y) .$$

Denote $H_1 = e\nu_f(A_x\sigma_x + A_y\sigma_y)$, where $\vec{A} = (A_x, A_y)$ is the vector potential of the light, and calculate the transition matrix element

$$M_{+-}(\vec{k}) = \langle \psi_+ | H_1 | \psi_- \rangle .$$

Assume that the vector potential \vec{A} is that to light with frequency ω , and the corresponding electric field given by $\vec{E}(t) = \vec{E}_0 e^{i\omega t}$.

c) Now let's assume that $E_-(\vec{k}) < E_+(\vec{k})$ and that the $E_-(\vec{k})$ states are always occupied while the $E_+(\vec{k})$ states are unoccupied. Use the Fermi's golden rule to calculate the optical transition rate Γ of Dirac electrons in graphene. Recall that the Fermi's golden rule is given by

$$\Gamma = \frac{2\pi}{\hbar} \int \frac{d^2k}{(2\pi)^2} |M_{+-}(\vec{k})|^2 \delta(E_+ - E_- - \hbar\omega) ,$$

where ω is the frequency of the light.

I-3. Two bodies have internal energy given by $U = NCT$. The number of particles N and the specific heat C are the same for each body; the initial temperatures of the two bodies are T_1 and T_2 . The two bodies are used to drive a Carnot engine which will bring them to a common final temperature T_f while doing work. The Carnot cycle, comprised of quasistatic isothermal transformations and two quasistatic adiabatic transformations, is sketched in figure I-3.1.

- a) Find the final temperature T_f .
- b) Find the work done by the engine.

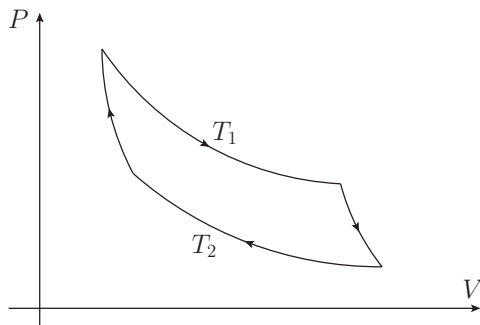


Figure I-3.1 The Carnot cycle.

I-4. Consider a parallel plate capacitor in vacuum, with circular plates each of radius a , separated by a distance d , see Figure I-4.1.

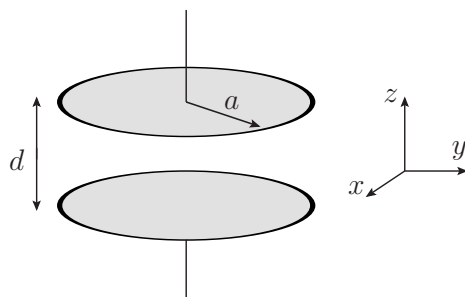


Figure I-4.1 Circular capacitor.

A current I is slowly charging the capacitor. Using the Poynting vector, defined in terms of the electric and magnetic fields \vec{E} and \vec{B} by

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

and representing the directional energy flux of an electromagnetic field, find the rate at which the electromagnetic field feeds energy into the capacitor. Show that the energy input is also equal to IV where V is the potential difference

between the plates. Assume that the electric field is uniform out to the edges of the plates.

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II-1. A relativistic proton collider consists of two storage rings, each with 100 *GeV* protons orbiting in the horizontal plane in the presence of nearly constant vertical magnetic fields. There is a slight displaced between these nearly circular orbits with one beam orbiting clockwise and the other counter-clockwise. Each ring has of radius $R_{\text{ring}} = 1 \text{ km}$ and contains 110 proton pulses (called bunches). At time $t = 0$, each bunch contains 10^{11} protons. At the time they collide the protons are randomly located within a cylindrical shape of length $L = 1 \text{ m}$ and radius $R = 0.1 \text{ mm}$. The cylinder axis is always concentric with the local beam axis. Bunch collisions only occur within a 1 m long interaction region along the ring circumference, where the circular beam orbits have been slightly perturbed so the bunches in this region move along a single common collision axis. A given bunch in one beam always collides with a particular bunch in the other beam. All proton collisions occur within the collision region.

At the interaction point, all protons in a beam bunch have equal momentum vectors parallel to the beam line and the opposite that of the other colliding beam bunch.

a) Absorption Cross section: Assume a proton passing through a bunch breaks up if its trajectory takes it to within a transverse distance of $d = 10^{-15} \text{ m}$ from a proton in the opposing bunch. What is the rate of interaction within a single bunch?

b) Assume the decay of the number of protons in a bunch is only due to these absorptive collisions. Derive the expression for the number of protons in a bunch as a function of time, $N(t)$ in terms of variables $(R_{\text{ring}}, R, L, d)$.

(Note: $N(0) = 10^{11}$ and that each bunch in one ring only collides with a unique partner bunch in the other ring, removing 1 proton from each of the partner bunches. Assume that the speed of protons is very near the speed of light.)

II-2. A particle is initially in its ground state in a box with infinite walls at $x = 0$ and $x = L$.

a) Find the probability density for finding the particle at position $x = L/2$.

a) The wall at $x = L$ is suddenly moved to $x = 2L$. Find the probability that the particle is found in the second excited state of the expanded box.

b) Suppose that the walls of the original box, $x \in [0, L]$, are suddenly dissolved while the particle was in its ground state. Find the probability that the freed particle has momentum p .

- II-3. Consider a rigid lattice of indistinguishable spin-1/2 atoms in a magnetic field H . Each spin has two states, with energies $+\mu_0 H$ and $-\mu_0 H$ for spins up and down, respectively. The system is at temperature T .
- Find the canonical partition function.
 - Determine the total induced magnetic moment of the system.
 - Determine the entropy of the system.

- II-4. A controlled voltage source, shown in Figure II-4.1, produces a voltage signal at output A . The source can be in one two states, shown in Figure II-4.2.

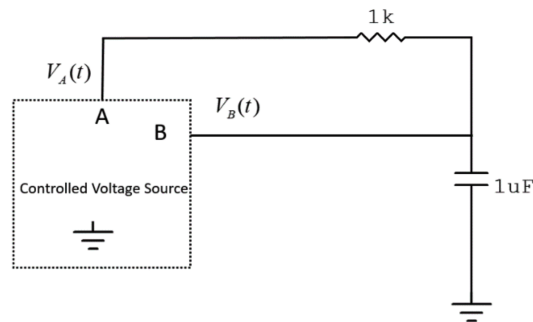


Figure II-4.1 Controlled voltage source.

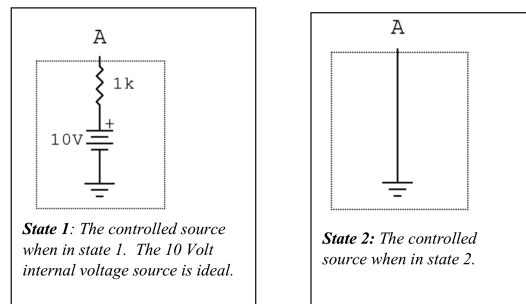


Figure II-4.2 The two possible states of the controlled voltage source.

When the voltage source is in state 1, the source provides 10 volts between output A and ground, with 1000 ohms of output impedance as shown. When the voltage source is in state 2, it provides a direct short to ground. The input B , which has infinite input impedance and is used to trigger a change of state.

- When the voltage at B increases and passes through 7 volts, the state is set to “state 2”.
- When the voltage at B is decreases and passes through 2 volts, the state is set to “state 1”.

- a) Determine the differential equation for the charge on the capacitor, $Q(t)$, when the source is in state 1.
- b) Determine the differential equation for $V_B(t)$ when the source is in state 2. Write down the general solution for $V_B(t)$.
- c) Determine the time for a full cycle state 1 to state 1.