Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 1st, 2020

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number on the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 GeV		$1.609 \times 10^{-10}\,\mathrm{J}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Infinite sums:

$$\sum_{n\geq 0} x^n = \frac{1}{1-x} \quad |x| < 1 \tag{I–3}$$

$$\sum_{n\geq 0} n x^n = \frac{x}{(1-x)^2} \quad |x| < 1 \tag{I–4}$$

$$\sum_{n\geq 0} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \quad |x| < 1 \tag{I–5}$$

Gradient in spherical polar coordinates (r, θ, ϕ):

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\,, \tag{I–6}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–7}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–8}$$

I–1. A particle of mass $m = 1$ is constrained to the plane (x, y) and is located at position $\vec{x}_0 = (2/\sqrt{3}, 0)$ at time $t = 0$, with velocity $\vec{v}_0 = (1/\sqrt{3}, \sqrt{3})$. It is subject to a force that results from the combined potentials $V_i(\vec{x}) = \frac{1}{4}|\vec{x} - \vec{R}_i|^2$, $i = 1, \ldots, 4$, where $\vec{R}_i$ are unit vectors pointing to the corners of a square with side length $\sqrt{2}$, centered at the origin and rotated by some angle α relative to the axes, see figure I-1.

(a) Is the angular momentum conserved in this system?

(b) Determine the minimal and maximal distance of the particle from the origin.

I–2. A simple harmonic oscillator of frequency ω is initially (at $t = 0$) in the state

$$\Psi(x, 0) = \mathcal{N} \sum_{n=0}^{\infty} c^n \psi_n(x)$$

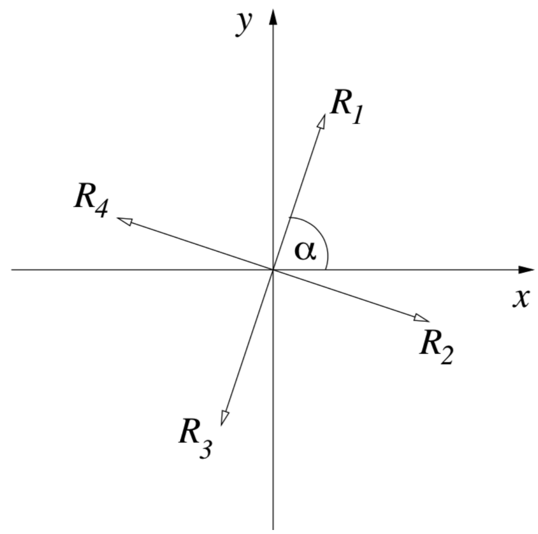

Figure I-1: Figure for problem I-1.

where $\psi_n(x)$ are the harmonic oscillator energy eigenstates corresponding to the n-th energy level and c is a free complex parameter with $|c| < 1$.

(a) Calculate the normalization constant $\mathcal{N}$

(b) Find the wave function of the system at a later time t

(c) Find the probability that the system is again in the initial state at a later time $t > 0$

(d) Compute the expectation value of the energy as a function of time

I–3. Consider a gas of N noninteracting quantum one-dimensional harmonic oscillators in equilibrium in a box of volume V at temperature T. The energy levels of a single oscillator are

$$E_m = (m + 1/2)\frac{\gamma}{V}$$

where γ is a constant.

(a) Find the entropy S of the system and the specific heat at constant volume C_V as a function of T.

(b) Determine the equation of state of the gas.

(c) Find the fraction of particles on the m-th energy level.

I–4. Two infinitely-large parallel grounded metal plates are placed at positions $-D/2$ and $+D/2$ on the x-axis, see figure I-4. A point charge Q is placed midway in between the plates. Using the method of images, find the surface charge density on the plate located at $+D/2$. If you encounter infinite sums, leave them unevaluated.

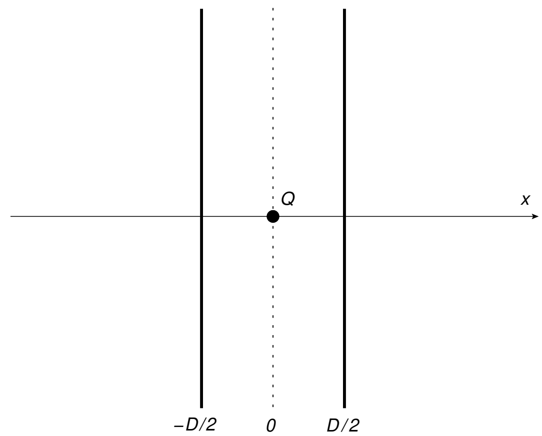

Figure I-4: Two infinitely-large parallel grounded metal plates are placed at positions $-D/2$ and $+D/2$ on the x-axis.

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II–1. A pendulum is constructed from a pointlike mass m suspended on an ideal string of length L. Its suspension point is at height $H < L$ above a horizontal plane. Assume that the plane is frictionless and is such that anything that collides with it instantly looses its momentum orthogonal to the plane.

(a) The pendulum is released from angle θ, see figure II-1. It undergoes a perfectly inelastic collision with a pointlike mass M placed on the plane, at angle $\theta_0 < \theta$ from the vertical through the suspension point. Find the energy and momentum of resulting particle immediately after the collision, assuming that the string breaks the instant before the collision.

(b) The particle resulting from the collision climbs on an inclined plane of height $h < H$ and angle α with friction coefficient μ. Find μ such that the particle stops precisely at the top of the inclined plane.

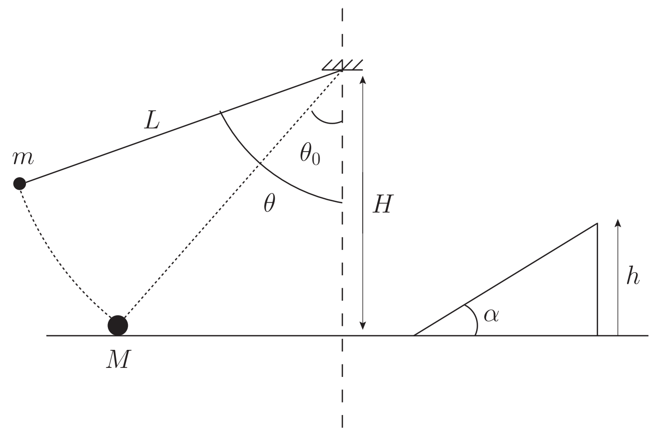

Figure II-1. Figure for problem II-1

II–2. (a) Compute the commutator $[\hat{\vec{x}} \cdot \hat{\vec{p}}, \hat{H}]$ where $\hat{H}$ is the Hamilton operator of the hydrogen atom.

(b) If ψ is a stationary state of the hydrogen atom, use the result of part (a) to show that the expected kinetic energy, $\langle \hat{T} \rangle = \langle \psi | \hat{T} | \psi \rangle$, is related to the expected potential energy, $\langle \hat{V} \rangle = \langle \psi | \hat{V} | \psi \rangle$, by $2\langle \hat{T} \rangle = -\langle \hat{V} \rangle$.

(c) Compute the expected kinetic and potential energies in the n-th energy eigenstate of hydrogen.

II–3. A box of volume $2V$ is divided into two halves by a thin wall. The left side contains an ideal gas at pressure p_0 and the right side is initially vacuum. A small hole of area A is punched in the dividing wall and the temperature is held constant and the same on both sides. Find the pressure on the left side of the divide as a function of time.

II–4. Consider two parallel long hollow cylindrical wires of radius a , separated by distance $d \gg 2a$, see figure II-4a. Assume that the voltage and surface currents are uniform over azimuth of each wire, but can vary over the length of the wire. The capacitance and inductance per length are

$$\rho_C = \frac{\pi \varepsilon}{\ln(d/a)} \qquad \rho_L = \frac{\mu}{\pi} \ln(d/a) \ ,$$

respectively. In vacuum $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$.

The pair of wires can be modeled electrically as a chain of small capacitors and inductors, with capacitors at locations $(\ldots, x - dx, x, x + dx, x + 2dx, \ldots)$ and inductors between capacitors, see figure II-4b. At a given position x, the current $I(x,t)$ in the upper wire is equal and opposite to the current in the lower wire. The voltage $V(x,t)$ is measured above the capacitor, see figure.

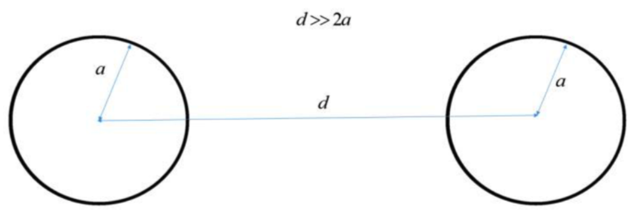

Figure II-4a: Cylindrical wires.

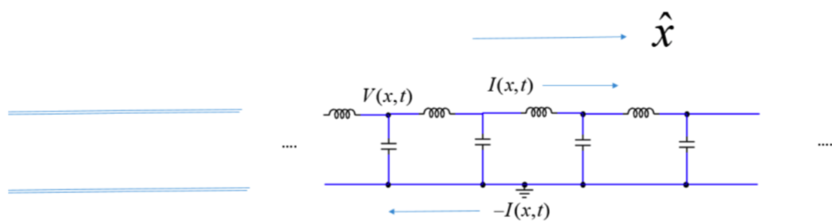

Figure II-4b: LC model for cylindrical wires.

(a) Derive an wave equation of the Voltage wave, $V(x,t)$.

(b) What is the speed of propagation in units of the speed of light?

(c) What would the speed of propagation be if the wires were surrounded by a region with dielectric constant $\kappa = 2$.