Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 2nd, 2019

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number on the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 GeV		$1.609 \times 10^{-10}\,\mathrm{J}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2+a^2)^{n-1}} n + c \quad \text{for } n \neq 0, 1 \tag{I–3}$$

Gradient in spherical polar coordinates (r, θ, ϕ):

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}\,, \tag{I–4}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}. \tag{I–5}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–6}$$

I–1. A projectile is launched nonrelativistically at an angle of 45 degrees with an initial kinetic energy E_0. At the top of the trajectory the projectile explodes into two fragments. The explosion imparts an additional mechanical energy E_0 to the system. One fragment, of mass m_1, travels straight down with an unknown velocity v_1. Assume that the motion is in the (x, y) plane.

(a) Find the two components of the velocity of the second fragment, of mass m_2, and the magnitude of the velocity of the first fragment

(b) What is the ratio of masses, m_1/m_2, that maximizes m_1.

I–2. Find the velocity of sound, v_s, for an ideal spin-1/2 Fermi gas at zero temperature. How does the sound velocity v_s compare to the Fermi velocity? [*Hint:* It may be useful to recall that $v_s^2 = B/\rho$, where the bulk modulus is $B = -V(\partial P/\partial V)_T$ and ρ is the mass density.]

I–3. Consider a particle of mass m in one dimension, subject to the potential

$$V(x) = \frac{\hbar^2}{2m}(\phi'^2(x) - \phi''(x))$$

where ϕ is an arbitrary real function and $\phi'(x)$ and $\phi''(x)$ are its first and second derivatives.

(a) Show that the Hamiltonian may be written as

$$H = \frac{1}{2m}(p + if(x))(p - if(x))$$

where f is a suitable real function.

(b) Can H have negative eigenvalues?

(c) Show that if $E = 0$ is an eigenvalue of H corresponding to a bound state $|E = 0\rangle$, then *necessarily*

$$(p - if(x))|E = 0\rangle = 0 \ .$$

(d) If the bound state $|E = 0\rangle$ exists, find its wave function $\psi_0(x)$.

(e) Can one claim the existence of the bound state with $E = 0$ for any function $\phi(x)$?

I–4. The difference of electrostatic potentials between two long coaxial cylindrical electrodes of inner radius a and outer radius b is V. A uniform magnetic field B is applied to the cylinders along their axis. If B is strong enough (greater than a critical value B_c) the electrons emitted with zero velocity from the cathode at the center cannot reach the anode. Find the strength of the critical magnetic field B_c. Express your answer in terms of the mass and charge of the electron (m and q_e), V, b, and a.

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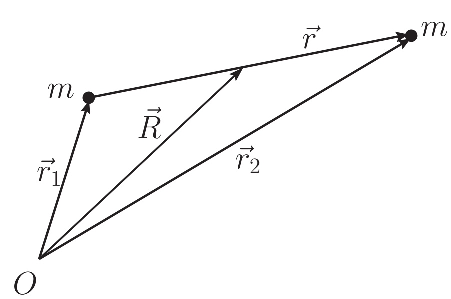
II–1. Two particles of equal masses, $m_1 = m_2 = m$, move on a frictionless horizontal surface near a fixed force center at the origin, with potential energy $U_1 = (1/2)kr_1^2$ and $U_2 = (1/2)kr_2^2$ where r_1 and r_2 are the distances of the particles from the force center. In addition they interact with each other via a potential energy $U_{12} = (1/2)\alpha k r^2$ where r is the distance between them and α and k are positive constants (see figure) .



(a) Write down the Lagrangian. Express it in terms of the center of mass position $\vec{R}$ and the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$.

(b) Write down the Lagrange equations for the center of mass position, $\vec{R} = (X, Y)$, and relative coordinates, $\vec{r} = (x, y)$.

(c) Solve the Lagrange equations for the above coordinates and describe the motion.

II–2. A large thin plate is placed in space in a plane normal to the line from the Sun to the plate. The plate is an ideal black body radiator. The plate is 1.5×10^{11} meters from the Sun and the Sun emits radiation with an average power of $3.85 \times 10^{26} W$.

(a) What is the steady state temperature of this single plate?

(b) If an identical second plate is located a short distance behind the first one, what will the temperature of the first and second plate be? Assume that the second plate is completely in the shadow of the first and that the only mechanism for heat transfer is radiation.

II–3. Use the probability current

$$\vec{j}(r, \theta, \phi) = \frac{\hbar}{2mi} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

to compute the magnetic dipole moment of a hydrogen atom in the state $\psi_{111}(r, \theta, \phi)$.

II–4. A sphere of radius R and uniform volume charge density ρ has an empty spherical cavity of radius a located at distance $R/2$ from the center of the sphere. The volume of the cavity is entirely within the volume of the sphere (*i.e.* $a < R/4$). Find the electric field everywhere inside the volume of the cavity. Remember to give both the magnitude and the direction. Assume that the center of the sphere of radius R is located at the origin and the center of the cavity lies along the z axis.