Qualifying Exam for Candidacy
Department of Physics
February 1, 2014
Part I

Instructions:

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• Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.

• We suggest you read all four of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

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Definite integrals:

\[
\int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]  

(I–1)

\[
\int_{0}^{\infty} x^n e^{-x} \, dx = \Gamma(n + 1) = n!.
\]  

(I–2)

Transformation of Lorentz 4-vector (e.g., \((ct, \mathbf{x})\)) under boost by velocity \(v\) in \(z\) direction:

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Laplacian in spherical polar coordinates \((r, \theta, \phi)\):

\[
\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.
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Laplacian in cylindrical coordinates \((r, \theta, z)\):

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\]  

(I–5)

I–1. A block of triangular cross section and of mass \(m_1\) is placed on a horizontal surface. On a diagonal face of the first block is placed another block, of mass \(m_2\). This face is at an angle of \(\alpha\) above the horizontal. The coefficients of dynamical friction on the top surface is \(\mu\), while the bottom surface is frictionless. Initially the blocks are held stationary, and at one particular time they are released. The conditions are such that there is sliding at both surfaces. Find the acceleration (including direction) of the first block (in terms of the masses, \(\mu\) and the gravitational acceleration \(g\)).

I–2. A thick spherical shell (inner radius \(a\), outer \(b\)) is made of dielectric material with a frozen-in polarization

\[
P = \frac{k}{r} \hat{r},
\]  

(I–6)

where \(k\) is a constant, \(r\) is the distance from the center (there is no free charge), and \(\hat{r}\) is the unit vector in the radial direction. Find the electric field everywhere.
I–3. Consider a two-state system with two observables, $A$ and $B$ each taking two values, $a_1$ and $a_2$ and $b_1$ and $b_2$, respectively. When $A$ takes value $a_i$ the normalized wave function of the system is $|\psi_i\rangle$ while when $B$ takes the value $b_i$ the normalized wave function of the system is $|\phi_i\rangle$. These wave functions are related to each other by
\[
|\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle, \quad |\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle. \tag{I–7}
\]

(a) The observable $A$ is measured and the value $a_1$ is obtained. What is the state of the system (immediately) after this measurement?
(b) If $B$ is now measured, what are the possible results, and what are their probabilities?
(c) Right after the measurement of $B$, $A$ is measured again. What is the probability of getting $a_1$? (Note that the outcome of the $B$ measurement is not specified.)

I–4. A spherical black body of radius $R_1$ is maintained at a constant absolute temperature $T$ by internal processes. It is surrounded by a thin spherical and concentric shell of radius $R_2$, black on both sides. The exterior temperature is $T_0$.

(a) Find the equilibrium temperature of the outer spherical shell.
(b) Find the ratio between the rate of energy loss in the presence and in the absence of the outer shell.
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\] (II–5)

II–1. Two protons of equal mass \(m_p = 0.938\, \text{GeV}/c^2\) and equal energy \(E_0 = 100\, \text{GeV}\) collide elastically. Denote the initial energies by \(E_{i1}\) and \(E_{i2}\), and the initial 3-momenta by \(\mathbf{p}_{i1}\) and \(\mathbf{p}_{i2}\). After the collision, the energies and 3-momenta are \(E_{f1}, E_{f2}, \mathbf{p}_{f1}\) and \(\mathbf{p}_{f2}\). The energies and momenta are given by
\[
E_{i1} = 100\, \text{GeV}, \quad \mathbf{p}_{i1} = p_0 \hat{z},
\] (II–6a)
\[
E_{i2} = 100\, \text{GeV}, \quad \mathbf{p}_{i2} = -p_0 \hat{z},
\] (II–6b)
\[
E_{f1} = 100\, \text{GeV}, \quad \mathbf{p}_{f1} = p_0 (\hat{x} \sin \theta + \hat{z} \cos \theta),
\] (II–6c)
\[
E_{f2} = 100\, \text{GeV}, \quad \mathbf{p}_{f2} = -p_0 (\hat{x} \sin \theta + \hat{z} \cos \theta).
\] (II–6d)

Assume the scattering angle \(\theta\) is \(0.01\) radians.

(a) Determine \(p_0\) in units of \(\text{GeV}/c\) and determine \(\gamma = 1/\sqrt{1-\beta^2}\) for the boost from the rest-frame of the proton. Here \(\beta = v/c\), with \(v\) being the speed of one proton in the center-of-mass reference frame.

(b) The same collision process is now observed in a reference frame where the target proton (2) is at rest before the collision. This is a frame boosted along the \(z\) axis from the original frame. What will the momenta of the two protons be after collision? Give \(x\) and \(z\) components for both final state protons.

(This is the transformation that describes a result from a collider experiment as seen in the frame common for fixed target experiments)
II–2. A planar circuit surrounds a solenoid and consists of two capacitors of capacitances $C_1$ and $C_2$ joined together by normal wires. The solenoid crosses the plane of the circuit in a patch of area $A$, and it produces a time-dependent magnetic field that is changing linearly with time: $B(t) = B_0 + \dot{B}t$; the positive direction is coming up out of the paper. The field is uniform inside the solenoid, and the return path for the flux is well outside the region shown on the picture, and the magnetic field outside the solenoid is to be neglected.

Before the field is applied the capacitors have zero charge. In equilibrium what are the charges $Q_1$ and $Q_2$ on the capacitors. Determine the signs.

![Diagram of a planar circuit surrounding a solenoid with capacitors $C_1$ and $C_2$.]

II–3. A quantum mechanical particle of mass $m$ in one dimension has the following square well potential energy:

$$V(x) = \begin{cases} 0 & \text{if } |x| \leq a, \\ V_0 & \text{if } |x| > a. \end{cases}$$  \hspace{1cm} \text{(II–7)}

Derive an equation whose solution gives the energy eigenvalue(s) for antisymmetric wave functions: $\phi(-x) = -\phi(x)$.

II–4. In 1906, J. B. Perrin started a series of experiments to determine Avogadro’s number, for which he was awarded the Nobel Prize in Physics in 1926. In those experiments, he used a microscope to measure the change in concentration of little spherical particles in water with the distance from the bottom of the container. The density of those particles (which he obtained from the resin called gamboge) was $\rho = 1.21 \times 10^3 \text{kg/m}^3$ and their volume $V = 1.03 \times 10^{-19} \text{m}^3$, while the density of water is $\rho_W = 1.00 \times 10^3 \text{kg/m}^3$. The experiment was done at a temperature $T = 4 ^\circ \text{C}$. Determine the distance from the bottom of the container at which the concentration of those particles halved.

II–3