Rigid Tilings of Quadrants by L-Shaped n-ominoes and Notched Rectangles[☆]

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Abstract

In this paper, we examine rigid tilings of the four quadrants in a Cartesian coordinate system by tiling sets consisting of L-shaped polyominoes and notched rectangles. The first tiling sets we consider consist of an L-shaped polyomino and a notched rectangle, appearing from the dissection of an $n \times n$ square, and of their symmetries about the first diagonal. In this case, a tiling of a quadrant is called rigid if it reduces to a tiling by $n \times n$ squares, each of the squares in turn tiled by an L-shaped polyomino and a notched rectangle. We further determine the rigidity of tilings of the quadrants with tiling sets appearing from similar dissections of $mn \times n$ rectangles. Our technique of proof is to use induction along a staircase line built out of $n \times n$ squares and to show that the existence of a tile in irregular position propagates further towards the edges of the quadrant eventually leading to a contradiction. Further generalizing, we examine sets of tiles appearing from dissections of rectangles of co-prime dimension into an L-shaped polyomino and a notched rectangle. These tilings are never rigid. We give descriptions of nonrigid tilings for each quadrant and for each tiling set of this type. Finally we record some general conjectures about problems of this type.

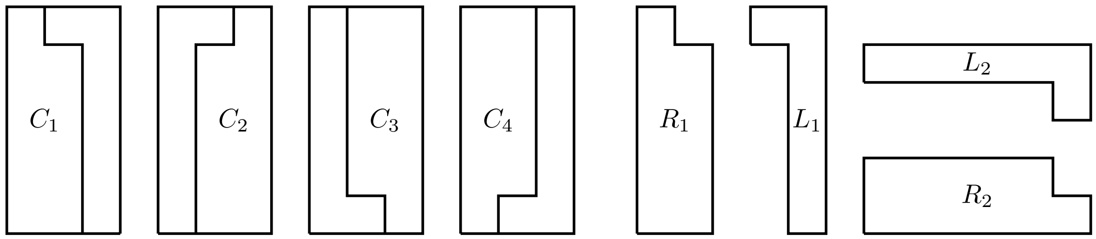
1. Introduction

In this article we study tiling problems for regions in a square lattice by certain sets of tiles consisting of polyominoes. Polyominoes were introduced by Solomon W. Golomb [2] and the standard reference about tiling with polyominoes is the book *Polyominoes* [4]. Here we continue the investigation started recently in the papers [1, 6], where the notion of *rigid tiling* is introduced for the tiling of the first quadrant by special tiling sets coming from the dissection of a rectangle in two L-shaped polyominoes and their symmetries about the first diagonal. A rigid tiling of the first quadrant is defined as one that reduces to a tiling of the quadrant by symmetries of the dissected rectangle, each tiled in turn by two pieces from the tiling set.

The 1×1 squares in the square lattice are called *cells*. We consider the four possible dissections of an $k \times n$ rectangle, $3 \leq n \leq k$, into an L-shaped polyomino and a notched rectangle. The missing part from the notched rectangle is always a cell, as exhibited below in Figure 1a, where $n = 3, k = 6$. We denote these dissections by C_1, C_2, C_3 and C_4. Each dissection determines two tiles from our tiling set. To complete the tiling set, we reflect the pieces resulting from the dissection about the first bisector. Only translations, that is, no rotations or reflections, of the tiles are allowed in a tiling. The L-shaped polyominoes are denoted L_1 and their reflections L_2, while the notched rectangles are denoted R_1 and their reflections R_2. The complete tiling set for the C_1 dissection is shown in Figure 1b. We denote the tiling set given by the dissection C_i by $\mathcal{T}(C_i, k, n)$.

Definition 1. *A tiling by $\mathcal{T}(C_i, k, n), 1 \leq i \leq 4, 3 \leq n \leq k$, of a region in the plane is called* rigid *if it reduces to a tiling by $k \times n$ and $n \times k$ rectangles, each tiled in turn by two pieces from the tiling set.*

[☆]This paper was written during the Summer 2013 program REU at Pennsylvania State University, supported by the NSF grant DMS-0943603. A. Calderon, S. Fairchild, M. Muir and S. Simon are undergraduate students. V. Nitica was one of the faculty coordinators. He was partially supported by Simons Foundation Grant 208729.



(a) The dissections C_1, C_2, C_3, C_4.

(b) A C_1 tiling set from the dissection of a 6×3 rectangle.

Figure 1: Our dissections and $\mathcal{T}(C_1, 6, 3)$.

The study of rigid tilings of an infinite quadrant started with the papers [1, 6], inspired by a problem from recreational mathematics [3, 5]. The dissections considered are those with rectangles of width 2 into two L-shaped polyominoes, not necessarily congruent. The tiling sets consist of four tiles: the two L-shapes generated by the dissection and their symmetries about the first bisector. When the height of the dissected rectangle is odd, the rigidity problem is completely solved by showing via counterexamples that the tiling set is never rigid. The situation is more complex when the height of the dissected rectangle is even. If the tiles appearing from the dissection are congruent, the problem has a complete solution. There are both rigid and nonrigid tiling sets appearing in infinite families. If the tiles are not congruent, several cases are solved, both rigid and nonrigid, and several cases are left open.

The discussion above explains the approach taken in this paper. While it is desirable to completely solve the rigidity problem for a quadrant and a tiling set given by a dissection of a rectangle in two polyominoes, it is difficult to do so even in the case of a rectangle of width 2. This is why we prefer to study special classes of rectangles/dissections that naturally generalize previous work and helps to clarify certain conjectures.

Our main results are:

Theorem 2. *1) Any tiling of the first quadrant by $\mathcal{T}(C_1, mn, n), n \geq 3, m \geq 1$, is rigid. There are nonrigid tilings of the other quadrants by $\mathcal{T}(C_1, mn, n), n \geq 3, m \geq 1$.*

2) Any tiling of the third quadrant by $\mathcal{T}(C_3, mn, n), n \geq 3, m \geq 1$, is rigid. There are nonrigid tilings of the other quadrants by $\mathcal{T}(C_3, mn, n), n \geq 3, m \geq 1$.

3) Any tiling of a quadrant by $\mathcal{T}(C_2, mn, n), n \geq 3, m \geq 1$, is rigid, except the case $n = 3, m = 1$ which is not rigid for the second, third and fourth quadrants.

4) Any tiling of a quadrant by $\mathcal{T}(C_4, mn, n), n \geq 3, m \geq 1$, is rigid, except the case $n = 3, m = 1$ which is not rigid for the second, third and fourth quadrants.

Several results from Theorem 2 are immediate due to the symmetries of the tiling sets. The tiling set $\mathcal{T}(C_3, k, n)$ can be obtained from $\mathcal{T}(C_1, k, n)$ via a rotation by 180°, and the tiling set $\mathcal{T}(C_4, k, n)$ can be obtained from $\mathcal{T}(C_2, k, n)$ via a rotation by 180°. Moreover, $\mathcal{T}(C_2, k, n)$ and $\mathcal{T}(C_4, k, n)$ are symmetric via a reflection about the second bisector. Thus to finish the proof of the rigidity results in Theorem 2 we only need to prove the corresponding part in 1) and 3). To prove the nonrigid results, it is enough to show a nonrigid tiling of the second and third quadrant by $\mathcal{T}(C_1, k, n)$.

Theorem 3. *If p, n are coprime, all quadrants have nonrigid tilings by $\mathcal{T}(C_i, p, n), 1 \leq i \leq 4$. More general, if p, n are co-prime, all quadrants have nonrigid tilings for tiling sets consisting of two L-shaped tiles, that appear from the dissection of a $p \times n$ rectangle, and from their symmetries about the first bisector.*

For the simplicity of the exposition, we only prove the first statement in Theorem 3. The proofs we provide can easily be carried over to the general case by the qualified reader.

Our results suggest the following conjectures for the general rigidity problem of an infinite quadrant:

- If one side of the dissected rectangle is a multiple of the other, then both rigid and nonrigid tilings are possible for infinite families of rectangles/dissections. The method of proof should be mathematical induction along a diagonal staircase line based on the $d \times d$ square lattice. Also, for infinite families of rectangles/dissections, the rigidity problem is difficult to solve, or even undecidable.

(a) Tiling the bottom left corner of the quadrant. (b) An example of the rigid pattern.

Figure 2: Rigidity of the bottom left $n \times n$ square.

- If the sides of the dissected rectangle have a proper common factor, then the problem is open. The first case of interest is the 6×4 rectangle, for which we do not have any results. Computer experiments suggest that a splitting in an L-shaped polyomino and a notched rectangle gives a rigid tiling.

Besides its intrinsic interest, the rigidity problem for tiling the first quadrant has numerous applications to tiling other regions in the plane such as rectangles, half-infinite strips, and double infinite strips. We refer to the papers [1, 6], where many such applications are derived. Several applications can be easily carried over to the cases studied in this paper. We will leave this task for the interested reader.

The rest of the paper is organized as follows: In Section 2 we show the proof of the rigidity result for $\mathcal{T}(C_1, n, n)$, and in Section 3 we show the proof of the rigidity result for $\mathcal{T}(C_2, n, n)$. We do this in order to outline the main ideas of the proof and in order to simplify the presentation of the general cases. In Section 4 we show the proof of the rigidity result for $\mathcal{T}(C_i, mn, n), m \geq 2, 1 \leq i \leq 4$, and in Section **??** we show the nonrigid results in Theorem 2. Theorem 3 is proved in Section 5.

2. Rigid Tiling of the First Quadrant by $\mathcal{T}(C_1, n, n)$.

In this section we prove the rigidity of tiling of the first quadrant by $\mathcal{T}(C_1, n, n), n \geq 3$. Besides the initial 1×1 square lattice, we use an $n \times n$ square lattice which has the same origin and same coordinate axes with the initial lattice. The squares in this lattice are called n-squares. We use induction on the k^{th} iteration of the $n \times n$ staircase line, as shown in Figure 3a. When we refer to an $n \times n$ staircase, here and in the future sections, we assume that all n-squares underneath the staircase line are already rigidly tiled. We draw the figures for $n = 3$, but the argument is general.

Consider the base case, $k = 1$. The cell in the corner of the first quadrant can only be covered by R_1 or R_2. Suppose we use R_1. Consider cell 1 in Figure 2a. If it is covered by R_1, R_2, or the long end of L_1, we are unable to tile cell 3, which is diagonally adjacent to cell 1. Likewise, if cell 1 is tiled by L_2 then cell 2, also diagonally adjacent to cell 1, is untileable. Thus the only possible tiling of cell 1 is with the short end of L_1, implying the bottom left n-square is rigidly tiled (see Figure 2b). If the cell in the corner is covered by R_2, rigidity follows from the previous argument by a reflection about the first bisector.

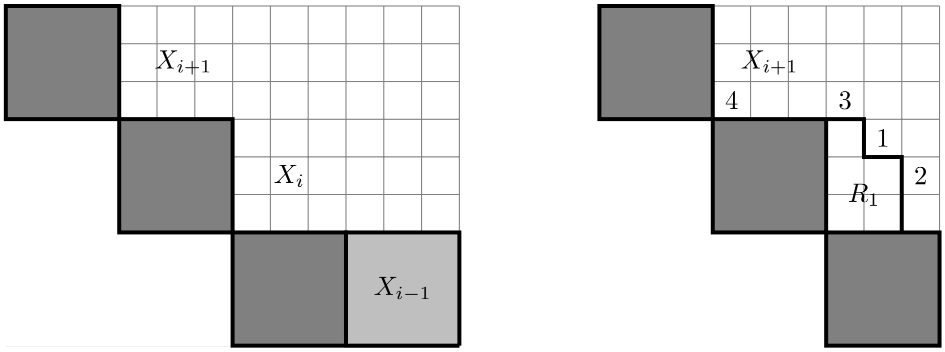
Now consider the staircase line for $k > 1$. Call the rightmost n-square above the staircase that does not follow the rigid pattern X_i (see Figure 3a). Look at what tile covers the bottom left cell of X_i. The square X_{i-1} to the lower right of X_i is rigidly tiled by our choice of X_i, so the covering of the corner cell by R_2 gives a rigid tiling of X_i via an argument identical to the base case. Assume then that the corner cell is tiled by R_1, as in Figure 3b. If we tile cell 1 by the short end of L_1, we again arrive at a rigid tiling of X_i. Thus we will consider all other possible tilings of cell 1.

We do not know if X_{i+1} is rigidly tiled, so we leave it open in the figures and address it as necessary.

Case 1: We use the long end of L_1 to tile cell 1. If the square X_{i+1} is already rigidly tiled, cell 3 cannot be tiled. Otherwise, the only possible tiling of cell 3 uses R_2 at cell 4, which leaves an untileable region to the left of the R_2 tile, or (if $n \geq 4$) uses L_2, which leaves an untilable region below the L_2 tile. Thus we cannot use the long end of L_1 to tile cell 1.

Case 2: If cell 1 is tiled with the long end of L_2, we are left with an untileable gap at cell 2, which is diagonally adjacent to cell 1.

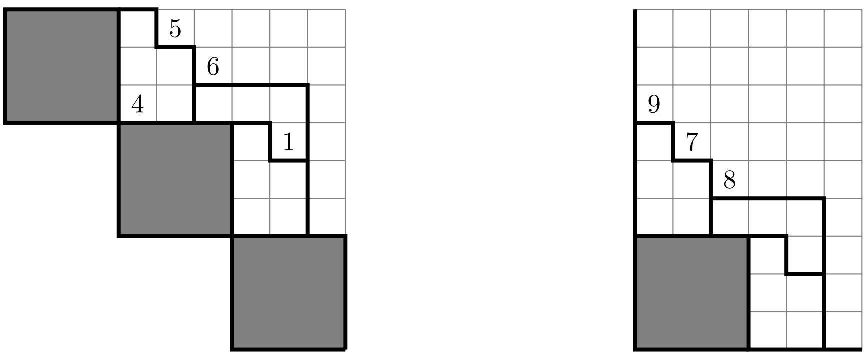
Case 3: Suppose cell 1 is tiled with the short end of L_2, as in Figure 4. Obviously square X_{i+1} cannot be previously rigidly tiled in this case. Then to tile cell 4 we are forced to use R_1. Now we must consider which tiles can be used to tile cell 5. See Figure 4a.



(a) The $n \times n$ staircase.

(b) Tiling the corner of X_i with R_1.

Figure 3: Attempts to tile X_i.



(a) Propagation of the pattern up the staircase.

(b) The end of the propagation.

Figure 4: Attempts to tile cell 1 with L_2.

- *Subcase 1* - If we attempt to use R_1, R_2 or the long end of L_2 to tile cell 5, then we are left with an untileable gap at cell 6, which is diagonally adjacent to cell 5. Thus we cannot tile cell 5 with R_1, R_2 or the long end of L_2.
- *Subcase 2* - If we attempt to use L_1 to tile cell 5, we are forced to use its long end. Then our subsequent argument follows as in Case 1.
- *Subcase 3* - Therefore, the only tile we may use to tile cell 5 is L_2. Continuing in this fashion, we must eventually reach the y-axis or a previously rigidly tiled square, see Figure 4b. Any attempt to tile cell 7 results in an untileable gap at one of the two cells diagonally adjacent to cell 7, labeled 8 or 9.

Therefore we cannot use the short end of L_2 to tile cell 1.

Case 4: We now attempt to tile cell 1 with R_1 or R_2. If X_{i+1} is already rigidly tiled, then cell 3 cannot be tiled. Otherwise, we may tile cell 4 with either R_1 or R_2. Note that tiling cell 4 by R_2 makes impossible to tile cell 3. Thus we are forced to tile cell 4 with R_1. Now the only way we can tile cell 3 and the cell on its immediate left is with R_1, as in Figure 5. If X_{i+2} is already rigidly tiled, we may not tile cell 5 without blocking off cell 10, so in the following subcases we will assume it has been left empty. Consider what tile covers cell 5.

- *Subcase 1* - If we tile cell 5 by L_2, then we reach the pattern of Case 3, which forces the creation of an untileable region.
- *Subcase 2* - Any attempt to tile cell 5 with L_1 implies the existence of R_2 to tile cell 10. As in Case 1, we are left with an untileable region.

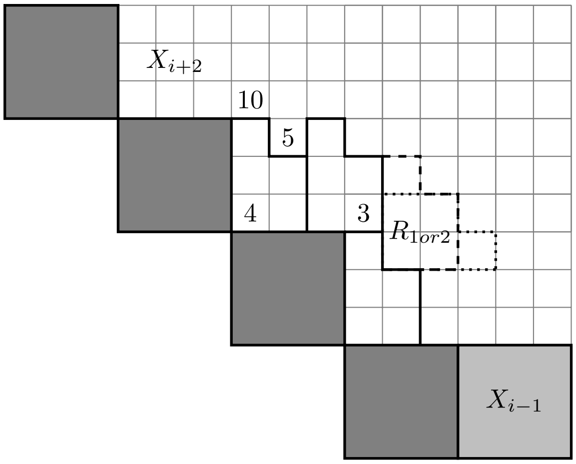


Figure 5: If we tile cell 1 by R_1 or R_2, we must tile cells 2 and 3 as shown.

Thus we cannot use R_1 or R_2 to tile cell 1.

Now we have shown cell 1 is untileable by the long end of L_1, the long or short ends of L_2, R_1 or by R_2, and so it must be tiled by the short end of L_1, see Figure 2b, and as such our pattern is rigid.

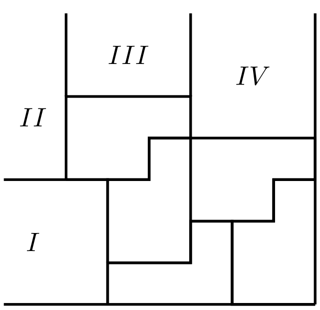
Therefore any tiling of the first quadrant by $\mathcal{T}(C_1, n, n)$ must be rigid.

3. Rigid Tilings of the Quadrants by $\mathcal{T}(C_2, n, n)$.

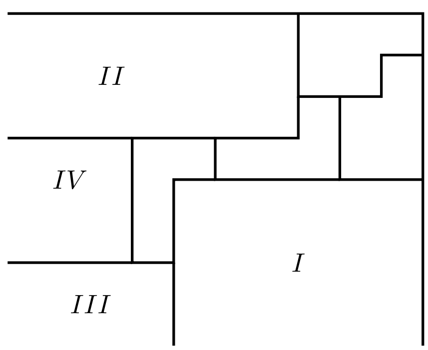
In this section we prove Theorem 2, 2), for the case $m = 1$.

In Figure 6a we show a nonrigid tiling of the second quadrant by $\mathcal{T}(C_2, 3, 3)$. Regions I, III and IV are half infinite strips of width 3 and region II is a copy of the second quadrant.

In Figure 6b we show a nonrigid tiling of the third quadrant by $\mathcal{T}(C_2, 3, 3)$. Regions II and IV are half infinite strips of width 3, region I is a half infinite strip of width 6 and region III is a copy of the second quadrant.



(a) A nonrigid tiling of the second quadrant.

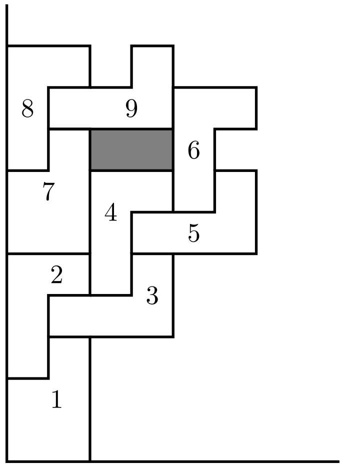


(b) A nonrigid tiling of the third quadrant.

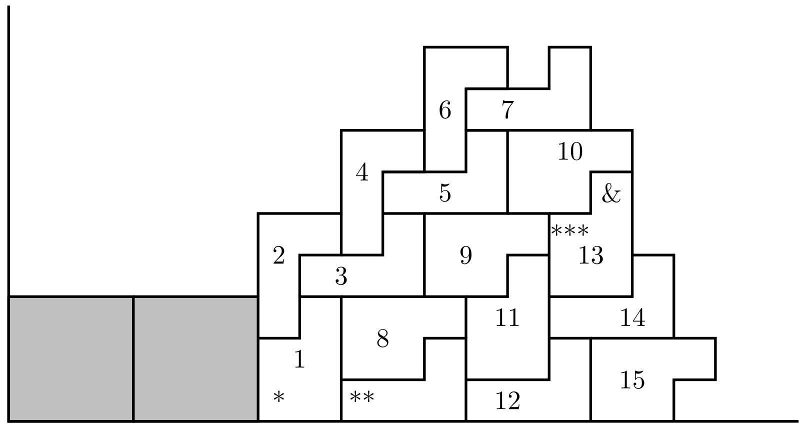
Figure 6: Non-rigid tilings by $\mathcal{T}(C_2, 3, 3)$.

We now prove rigidity of a tiling of the first quadrant by $\mathcal{T}(C_2, 3, 3)$. We will prove by induction that for any tiling of the first quadrant, the bottom row of 3-squares in the quadrant is tiled following the rigid pattern.

We consider the base case, $k = 1$. If R_1 or R_2 cover the corner cell, then a nontilable region is forced to appear as illustrated in Figure 7a (for R_2 use a symmetry about the main bisector). The tiles are labeled in the order in which we are forced to place them. If L_1 or L_2 cover the corner cell, then either the tiling of the 3-square in the corner is rigid, or we are forced into one of the configurations appearing in Figure 7a.



(a) R_1 covers the corner cell.



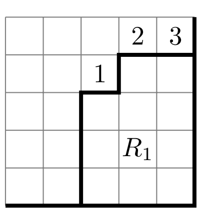
(b) General step of induction.

Figure 7: Tiling first quadrant by $\mathcal{T}(C_2, 3, 3)$.

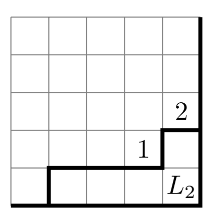
We assume now that some 3-squares in the bottom row are tiled following the rigid pattern and we look at the first 3-square for which the tiling is unknown. See Figure 7b. If cell * in Figure 7b is tiled by L_1 or R_2 we are leaded to a contradiction identical (via a symmetry about the first bisector) to that appearing in Figure 7a. Assume now that cell * is tiled by R_1. See Figure 7b. A staircase pointing upward is forced to appear determined by the tiles 1 through 7. We look now at cell **. If covered by L_1 or R_2, we are leaded to a contradiction identical to that appearing in Figure 7a. Tiling cell ** by R_1 leads to a cell that cannot be tiled, so cell ** has to be tiled by L_2. This leads to a forced tiling that eventually leads to a contradiction identical to that appearing in Figure 7a. See Figures 7b in which tiles 9 through 15 are forced. To see why tile 13 is forced, look at all possible tilings of cell ***, and then of the cell &. Moving forward with the tiling after tile 15 leads to a contradiction as in Figure 7a.

Finally, assume that cell * in Figure 7b is tiled by L_2. If the tile is part of a rigid tiling, we are done. Otherwise there is an R_1 tile above the L_2 tile, as in Figure **??**. We look at cell *. If tiled by R_1, this forces an R_2 tile below the R_1, and then leaded to a contradiction identical to that appearing in Figure 7a.

We now prove the rigidity of every tiling of the second quadrant by $\mathcal{T}(C_2, n, n), n \geq 4$, using induction on the staircase line. Consider our base case, $k = 1$. The cell in the corner of the second quadrant can only be covered with R_1 or L_2. Observe that if it is tiled by R_1, then any tile other than L_1 that is used to tile cell 1 makes it such that we cannot tile both cells 2 and 3. Similarly, if we use L_2 to tile the corner cell, any tile used to cover cell 1 in Figure 8b other than R_2 makes it impossible to tile cell 2. Thus the bottom right n-square must be rigidly tiled.



(a) A tiling of the corner cell by R_1.



(b) A tiling of the corner cell by L_2.

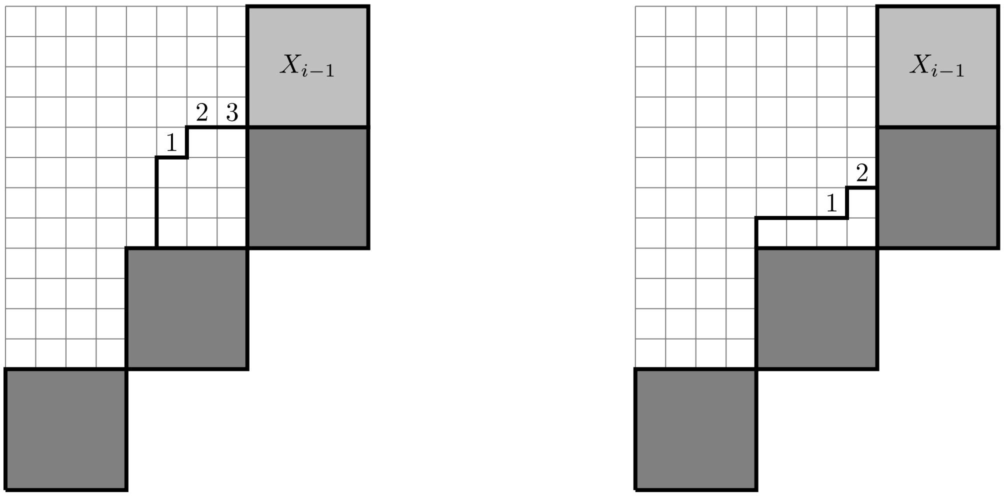
Figure 8: Any tiling of the corner cell implies the corner 4-square is rigidly tiled.

Now consider the staircase line for $k > 1$. Call the topmost n-square above the staircase that does not follow the rigid pattern X_i, similarly to what was done for the first quadrant. Look at what tile covers the bottom right corner cell of X_i (Figure 9). We again assume that the n-square X_{i-1} to the top right of X_i is rigidly tiled and that we are unaware whether square X_{i+1} is rigidly tiled.

Case 1: Suppose the bottom right corner of X_i is tiled with R_1, as in Figure 9a. Then, by the same argument we used in our base case, no tile may cover cell 1 other than L_1.

Case 2: Then the corner cell is covered by L_2, see Figure 9b. Just as in the base case, any attempt to tile cell 1 other than with R_2 leaves cell 2 untileable.

So X_i must be rigidly tiled, and thus any tiling of the second quadrant by $\mathcal{T}(C_2, n, n)$, $n \geq 4$, is rigid.



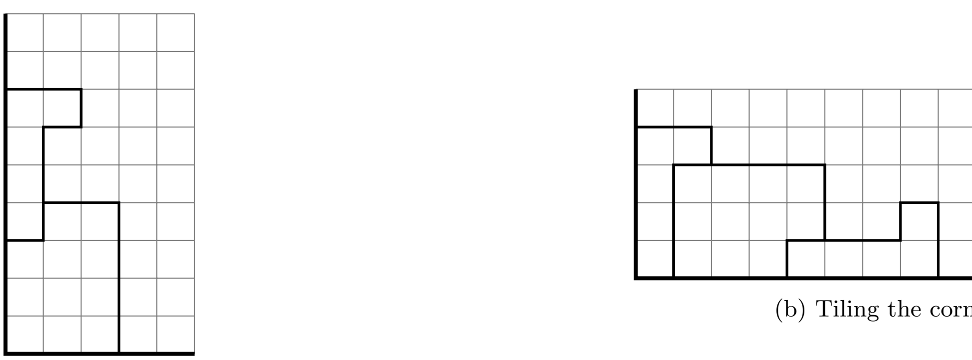
(a) Tiling the corner of X_i with R_1.

(b) The result if we use L_2 to tile the corner.

Figure 9: Attempts to tile X_i.

To prove rigidity in the first and third quadrants for $\mathcal{T}(C_2, n, n)$ and $\mathcal{T}(C_4, n, n)$, $n \geq 4$, note that it suffices to prove that $\mathcal{T}(C_2, n, n)$, $n \geq 4$ is rigid in these quadrants. $\mathcal{T}(C_4, n, n)$ can be derived from $\mathcal{T}(C_2, n, n)$ by reflection over the second bisector, so our argument for C_2 in the third quadrant is the same as the one for C_4 in the first, and vice-versa.

We first consider tilings of the first quadrant by $\mathcal{T}(C_2, n, n)$, $n \geq 4$. If the bottom left corner is tiled by R_1, we have a cell that can only be covered by L_1. See Figure 10a. But then we are left with a $(n-2) \times 1$ region to the right of the L_1 tile that cannot be covered by any tile in $\mathcal{T}(C_2, n, n)$. Similarly, we may not tile the corner by R_2. If we use L_1 to tile the corner, then we have a $(n-1) \times 1$ region which can only be tiled by R_2. But then we are forced to tile the unit cell as before, creating the same untileable region. See Figure 10b. Therefore the bottom left corner must be rigidly tiled.



(a) Tiling the corner cell by R_1.

(b) Tiling the corner cell by L_1.

Figure 10: Tiling the corner cell of the first quadrant.

Now consider the $k - th$ iteration of the staircase line. Call the furthest right nonrigidly tiled n-square X_i. If we attempt to tile its corner with L_1, as we assume X_{i-1} is rigidly tiled, then the same logic holds as in the bottom left corner. Tiling the cell with R_1 or R_2 leaves a single untiled cell which can only be filled as in the corner cases. We are then left with the same untileable regions. Thus we must use L_2 if we are to tile X_i nonrigidly. The $1 \times (n-1)$ region directly above the L_2 tile must then be tiled with an R_1 tile (11)

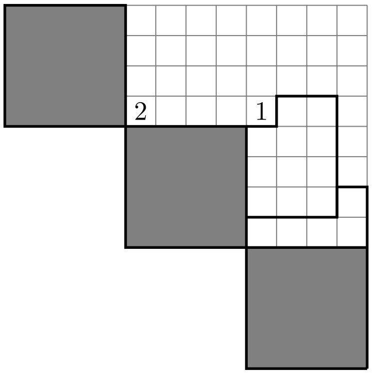


Figure 11: Tiling X_i on the staircase line.

Now note that if X_{i+1} is already rigidly tiled, then we cannot tile cell 1 except as in the corner, leading to a region which we may not tile. As such, we must assume X_{i+1} is not already rigidly tiled. Considering what tile covers the corner of that n-square (cell 2) we see that any possible covering of the cell results in the creation of an untileable region. So X_i must be rigidly tiled, and we have proved that any tiling of the first quadrant with the $\mathcal{T}(C_2, n, n), n \geq 4$ is rigid.

We now attempt to tile the third quadrant with $\mathcal{T}(C_2, n, n)$, $n \geq 4$. Note that we cannot tile the upper right corner with L_1 or L_2, as we would leave untileable regions next to the axes (12a). So we attempt to use R_1 or R_2 as before. The same reasoning works for both as they are reflections over the first bisector. But then we are left with cell 1 in Figure 12b, that we cannot tile except with L_1. So the top right n-square must be rigidly tiled.

Consider now the staircase line in the third quadrant. We define the staircase just as for the first quadrant, but we rotate it 180°. Consider again the furthest left nonrigidly tiled square. Its top right corner cannot be tiled by L_1 or L_2 by the same reasons as the top right corner of the quadrant, and for the same reason if we cover it with R_1 or R_2 it must be rigidly tiled. So any tiling of the third quadrant by $\mathcal{T}(C_2, n, n)$, $n \geq 4$ must be rigid.

4. Rigidity Results for $\mathcal{T}(C_i, mn, n)$

In this section we finish the proofs of Theorem 2, 1)-2), so we consider tilings by $\mathcal{T}(C_i, mn, n)$ where $1 \leq i \leq 2, m \geq 2, n \geq 3$. The diagrams are drawn for $n = 3, m = 2$, but the argument is general.

Definition 4. *A gap is an $n \times k$ region, $k > 0$, region such that the both the x and y coordinates of the corner closest to the origin are both divisible by n. We call k the length of the gap.*

By convention, we may assume that all n-squares that fall in the intersection of the top boundary of the gap, the right side of the gap and the x and y axes while remaining outside the gap itself are rigidly tiled.

(a) Top right corner by L_1.

(b) Top right corner by R_1.

Figure 12: Tiling the top right corner by a C_2-dissected n-square.

Definition 5. *We say an L_1 tile is in an irregular position if the corner closest to the origin has both its x and y coordinates divisible by n and if all $n \times n$ squares bound by this corner and the axes are rigidly tiled.*

Lemma 6. *Any nonrigidly tiled gap of the i^{th} quadrant by $\mathcal{T}(C_i, mn, n)$ induces an L_1 tile in an irregular position or a nonrigidly tiled gap closer to the y-axis.*

Proof: Choose the gaps that are closest to the x-axis, and with respect to those choose the gap closest to the y-axis. First, we consider the case of the first quadrant and $\mathcal{T}(C_1, mn, n)$. Consider the bottom left corner of the gap. It can be tiled by either L_1, R_1 or R_2.
Case 1: If it is tiled by L_1, then we are done as L_1 is placed in an irregular position.
Case 2: If it is tiled by R_2 as in Figure 13a, then we must view what tile covers cell 1.

- *Subcase 1:* If cell 1 is tiled by L_1, we have found our irregular tile.
- *Subcase 2:* If cell 1 is tiled by R_1 or R_2, it implies that cell 2 must be tiled by either L_2 which is a contradiction because we assumed nonrigidity of the gap, or by L_1, which forces L_2 to tile cell 3, creating an untilable region under the L_2 tile..

Case 3: Suppose the bottom left corner is tiled by R_1 as in Figure 13b. Now consider the possibilities to tile cell 1.

- *Subcase 1:* Suppose cell 1 is already rigidly tiled, then the entire n-square must be rigidly tiled. So, either cell 2 must be tiled by an L_1 tile in contradiction with our supposition that the gap was nonrigidly tiled or cell 2 must be covered by L_2. If cell 2 is covered by L_2, it is then impossible to tile cell 3.
- *Subcase 2:* If cell 1 is tiled by L_1, then we are done because it is in an irregular position.
- *Subcase 3:* If cell 1 is tiled by R_1 or R_2 such that none of the tile is above cell 2, then there will be a nonrigidly tiled gap closer to the y-axis. If any portion of the tile is above cell 2, then it implies that cell 2 must be tiled by either L_1 which is a contradiction because we assumed nonrigidity of the gap, or by L_2, which makes cell 3 impossible to tile.
- *Subcase 4:* If it is tiled by the left end of L_2, then cell 2 must be tiled with L_1, so it is rigid.
- *Subcase 5:* If cell 1 is tiled by any other part of L_2, then there is a nonrigidly tiled gap closer to the y-axis.

Now we consider the case for the second quadrant and $\mathcal{T}(C_2, mn, n)$. Consider the bottom right corner of the gap. It can be tiled by either L_1, R_1 or L_2.
Case 1: If it is tiled by L_1, then we are done as L_1 is placed in an irregular position.
Case 2: If it is tiled by L_2 as in Figure 14a, then we must view what tile covers cell 1.

- *Subcase 1:* If cell 1 is tiled by L_1, then there is a nonrigid gap closer to the y-axis.
- *Subcase 2:* If cell 1 is tiled by R_1 or L_2, then cell 2 cannot be tiled.
- *Subcase 3:* If cell 1 is tiled by R_2, then either the area is rigid in contradiction to the supposition, or we are unable to tile cell 2.

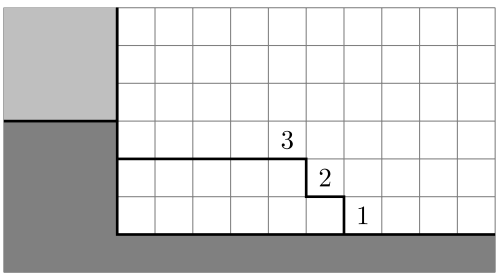
Case 3: Suppose the bottom right corner is tiled by R_1 as in Figure 13b. Now consider the possibilities to tile cell 1.

- *Subcase 1:* Suppose cell 1 is already rigidly tiled, then the entire n-square must be rigidly tiled. So, either cell 2 must be tiled by an L_1 tile in contradiction with our supposition that the gap was nonrigidly tiled or cell 2 must be covered by R_2. If cell 2 is covered by R_2, then either it is then impossible to tile cell 3 and the cells below it simultaneously for m odd, or we can only do so with copies of L_2, for m even, in which case we create untileable gaps between the L_2 tiles.
- *Subcase 2:* If cell 1 is tiled by L_1, then we are done because it is in an irregular position.

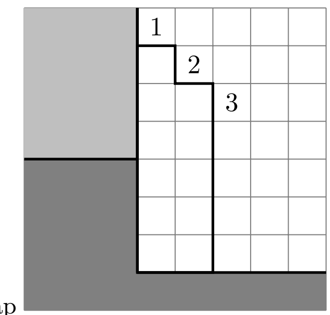
- *Subcase 3:* If cell 1 is tiled by R_1, R_2, or L_2 such that none of the tile is above cell 2, then there will be a nonrigidly tiled gap closer to the y-axis. If any portion of the tile is above cell 2, then it implies that cell 2 must be tiled by either L_1 which is a contradiction because we assumed nonrigidity of the gap, or by R_2. The argument then follows as in Subcase 1.

- *Subcase 4:* If R_2 covers both cells 1 and 2 (this only happens if $n = 3$), then cell 4 is covered by either L_2, which immediately induces and untileable region above the L_2 tile, or by R_2, in which case cell 3 is covered by L_2. Continuing this process as necessary, we are left with a region of height 2 which cannot be tiled.

Therefore, any nonrigid tiling of an $n \times k$ gap forces either a contradiction, an L_1 tile in an irregular position, or a nonrigidly tiled gap closer to the y-axis.

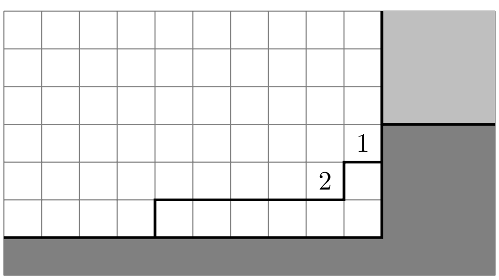


(a) An attempt to tile the bottom left corner of the gap with R_2.

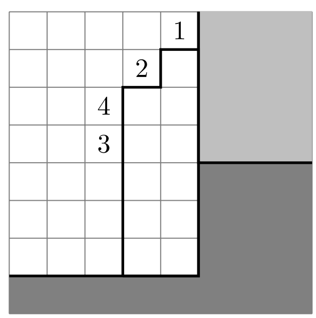


(b) Using R_1 to tile the bottom left corner of the gap.

Figure 13: Covering the bottom left corner of the gap.



(a) Using L_2 to tile the bottom right corner of the gap.



(b) Using R_1 to tile the bottom right corner of the gap.

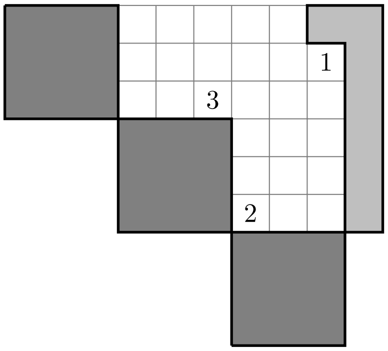
Figure 14: Possible tilings of the bottom right corner of a gap.

Lemma 7. *Tilings of the i^{th} quadrant by $\mathcal{T}(C_i, mn, n)$ cannot contain an L_1 in an irregular position.*

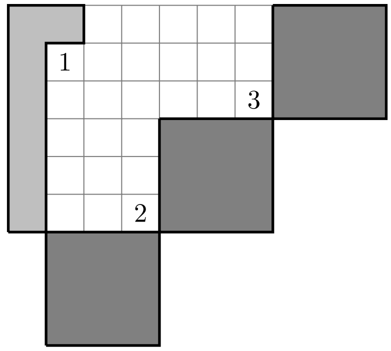
Proof: *Case 1*: First we look at the case of $\mathcal{T}(C_1, mn, n)$ in the first quadrant. Assume that we place an L_1 in an irregular posistion as in Figure 15a. We then consider cell 1. L_1 cannot cover 1. If R_1 tiles 1, then L_1 must tile cell 2, but then cell 3 cannot be tiled. If R_2 tiles square 1, then L_2 must cover cell 3, and the $n \times n$ region just above the rigidly tiled section cannot be tiled. If L_2 tiles 1, then it is impossible to tile cell 3. *Case 2*: We now look at the case of $\mathcal{T}(C_2, mn, n)$ in the second quadrant. Assume that we place an L_1 in an irregular position as in Figure 15b. We then consider cell 1. L_1 cannot cover 1. If R_1 tiles 1, then L_1 must cover cell 2, and the $n \times n$ region to the left of the rigidly tiles region cannot be tiled. If R_2 covers cell 1, then cell 3 cannot be covered. If L_2 covers cell 1, then cell 3 still cannot be covered.

Thus for tilings of the n^{th} quadrant by C_n, there cannot be an L_1 tile in an irregular position.

Theorem 8. *Any tiling of the i^{th} quadrant by $\mathcal{T}(C_i, mn, n)$ is rigid.*



(a) An irregular L_1 tile in the first quadrant.



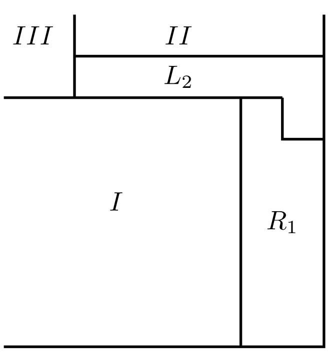
(b) Placement of an irregular L_1 tile in the second quadrant.

Figure 15: Irregular L_1 tiles in the first and second quadrants, respectively.

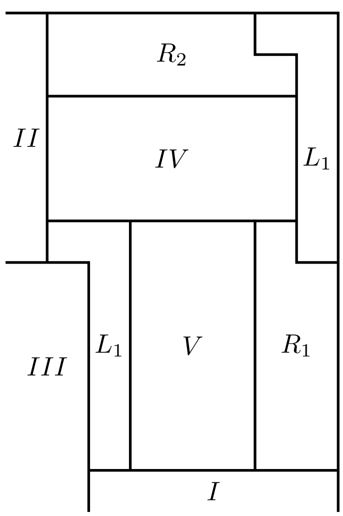
Proof: Assume for the sake of contradiction that there is a nonrigid tiling of the i^{th} quadrant. Then there exists a region that is nonrigidly tiled, which implies that there is an n-square that is nonrigidly tiled and hence there is a nonrigid gap. By Lemma 6, this implies there must either be an L_1 in an irregular position or a nonrigid gap closer to the y-axis. This gap in turn either induces an L_1 tile in an irregular position or a gap closer to the y-axis. However, this logic continues until the gap borders the y-axis which forces an L_1 tile in an irregular position since a gap cannot exist any closer to the y-axis. Then since we will have an L_1 tile in an irregular position, by Lemma 7, we know that a tiling of the first quadrant does not contain an irregular L_1 tile. Hence, any tiling of the i^{th} quadrant by $\mathcal{T}(C_i, mn, n)$ is rigid.

5. Non Rigid Results for $\mathcal{T}(C_1, mn, n)$.

As mentioned in the Introduction, in order to prove the nonrigid results in Theorem 2 it is enough to show a nonrigid tiling of the second and third quadrant by $\mathcal{T}(C_1, mn, n)$. We will use repeatedly in the proofs that a multiple of a rectangle of size $mn \times n$ or $n \times mn$, or an half-infinite strip of width a multiple of n or p can be tiled by the tiling set.

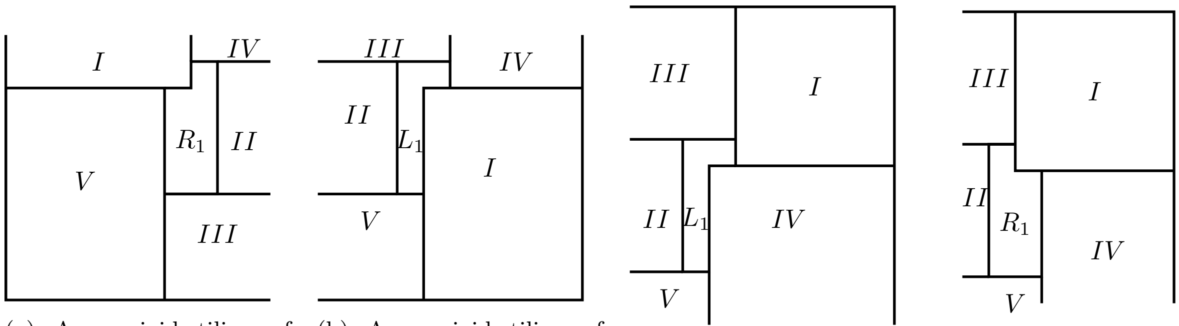


(a) A nonrigid tiling of the second quadrant by $\mathcal{T}(C_1, mn, n)$.



(b) A nonrigid tiling of the third quadrant by $\mathcal{T}(C_1, mn, n)$.

Figure 16



(a) A nonrigid tiling of the first quadrant by $\mathcal{T}(C_2, p, n)$.

(b) A nonrigid tiling of the second quadrant by $\mathcal{T}(C_2, p, n)$.

(c) A nonrigid tiling of the third quadrant by $\mathcal{T}(C_2, p, n)$.

(d) A nonrigid tiling of the third quadrant by $\mathcal{T}(C_1, p, n)$.

Figure 17: Non-rigid tilings by $\mathcal{T}(C_1, p, n)$ and $\mathcal{T}(C_2, p, n)$ for n, p coprime.

In Figure 16a we show a nonrigid tiling of the second quadrant by $\mathcal{T}(C_1, mn, n)$. Regions I and II are infinite strips of width mn and region III is a copy of the second quadrant.

In Figure 16b we show a nonrigid tiling of the third quadrant by $\mathcal{T}(C_1, mn, n)$. Regions I and II are infinite strips of width mn, region III is a copy of the second quadrant, region IV is a rectangle of size $(m-1)n \times mn$ and region V is a rectangle of size $mn \times (m-1)n$.

6. Non Rigid Results for Coprime Dimensions

In this section we prove Theorem 3. Due to the symmetries present in the tiling sets, it is enough to show the following:

1. nonrigid tilings for $\mathcal{T}(C_1, p, n)$ for the first, second and third quadrants;
2. nonrigid tilings for $\mathcal{T}(C_2, p, n)$ for the first, second and third quadrants.

We first give a proof for 2. and then give a proof of 1.

We will use that a multiple of an $p \times n$ or $n \times p$ rectangle, or a half infinite strip of width p or n can be tiled by $\mathcal{T}(C_i, p, n), 1 \leq i \leq 2$. Recall that if p, n are coprime, there exists positive integers x, y such that $yp - xn = 1$ (use for example that p is the generator of $\mathbb{Z}/n\mathbb{Z}$).

2. In Figure 17a we show a nonrigid tiling of the first quadrant by $\mathcal{T}(C_2, p, n)$ where p, n are coprime. For $x, y \in \mathbb{N}$ such that $xp - yn = 1$, region I is an infinite strips of width xp, region II is an infinite strip of width p, region III is an infinite strip of width $(p-1)yn$, region IV is a copy of the first quadrant and region V is a rectangle of size $(p-1)px \times yn$.

In Figure 17b we show a nonrigid tiling of the second quadrant by $\mathcal{T}(C_2, p, n)$ where p, n are coprime. For $x, y \in \mathbb{N}$ such that $xp - yn = 1$, region I is a rectangle of size $yn \times xp$, region II is an infinite strip of width p, region III is a copy of the second quadrant, region IV is an infinite strip of width yn and region V is an infinite strips of width $(x-1)p$.

In Figure 17c we show a nonrigid tiling of the third quadrant by $\mathcal{T}(C_2, p, n)$ where p, n are coprime. For $x, y \in \mathbb{N}$ such that $xp - yn = 1$, region I is a rectangle of size $xp \times yn$, region II is an infinite strip of width p, region III is an infinite strip of width yn, region IV is an infinite strips of width xp and region V is a copy of the third quadrant.

1. Observe that the tile L_2 in $\mathcal{T}(C_1, p, n)$ can be obtained via a counterclockwise rotation by 90° from L_1 in $\mathcal{T}(C_2, p, n)$. Using 2., this gives nonrigid tilings by $\mathcal{T}(C_1, p, n)$ for the first and second quadrants.

In Figure 17d we show a nonrigid tiling of the third quadrant by $\mathcal{T}(C_1, p, n)$ where p, n are coprime. For $x, y \in \mathbb{N}$ such that $xn - yp = 1$, region I is a rectangle of size $xn \times (n+2)yp$, region II is an infinite strip of width p, region III is an infinite strip of width yp, region IV is an infinite strip of width kxn and region V is a copy of the third quadrant.

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