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SEMIGROUP CONJECTURES FOR CENTRAL SEMIDIRECT PRODUCT OF $\mathbb{R}^n$ WITH $\mathbb{R}^m$

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Abstract. In this paper we prove two new results about closed semigroups in the family of solvable groups

$$H_{mn} := \mathbb{R}^m \ltimes_\phi \mathbb{R}^n$$

where ϕ, the structure homomorphism, maps nontrivially into the center of $\mathrm{Aut}(\mathbb{R}^n)$. The first result states that the closure of a semigroup generated by a set in H_{mn} that is not included in a maximal semigroup with nonempty interior is actually a group. The second result states that among the sets in H_{mn} that are not included in a maximal proper semigroup, those that generate H_{mn} as a closed semigroup are dense. Results of this nature were obtained before only for extensions of nilpotent groups. An application of our results is to the question of generic topological transitivity of skew-extensions of a hyperbolic system with fiber H_{mn}.

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1. Introduction

Given a continuous transformation $f : \mathcal{X} \to \mathcal{X}$, a Lie group G, and a continuous map $\beta : \mathcal{X} \to G$ called a *cocycle*, one can define a *skew product with fiber* G by $f_\beta : \mathcal{X} \times G \to \mathcal{X} \times G, f_\beta(x, \gamma) = (fx, \gamma\beta(x))$. We recall that f_β is *(topologically) transitive* if it has a dense forward orbit $\{f_\beta^n(x_0) | \ n \in \mathbb{N}\}$. An interesting dynamics problem is whether Lie group extensions of a hyperbolic basic set are typically topologically transitive.

One can find in [1] a general conjecture about transitivity: modulo the obstruction that the range of the cocycle is included in a maximal semigroup with nonempty interior, the set of C^r transitive cocycles contains an open and dense subset. The conjecture is proved for various classes of Lie groups G that are semidirect products of compact and nilpotent in [1, 2, 3, 4, 5, 6, 7].

Our goal is to focus on a related semigroup problem. The proof of transitivity of f_β in [1] is based on showing that the set $\mathcal{L}_\beta(x)$ of heights of β over a periodic point x is the whole fiber G. To obtain that $\mathcal{L}_\beta(x) = G$, one has to prove that for a typical family $F \in G^p$ that generates G as a group, if F is not contained in a maximal semigroup with nonempty interior, then F generates G as a semigroup as well. We refer to this question as the *Semigroup Problem*. The problem was solved for $G = \mathbb{R}^n$ [6, Lemma 5] and more generally $G = K \times \mathbb{R}^n$ where K is a compact Lie group [1, Theorem 5.10]. It is also solved for $G = SE(n)$ [1, Theorem 6.8] and for Heisenberg groups in [7, Theorem 8.6].

In this paper, we solve the Semigroup Problem for a family of solvable groups related to the group of affine transformations of the line Aff^+, which is the simplest non-nilpotent solvable group and the unique simply connected, nonabelian, 2-dimensional group. The groups under consideration are semidirect products $\mathbb{R}^m \ltimes_\phi \mathbb{R}^n$, where the structure homomorphism maps into the center of $\mathrm{Aut}(\mathbb{R}^n)$, the set of positive scalar matrices. To simplify the notation, we denote these groups as H_{mn}.

The structure of the maximal semigroups with nonempty interior in a solvable Lie group is described in a fundamental paper by Lawson [8]. They are in one-to-one correspondence, via the exponential map, with closed half-spaces with boundary a Lie subalgebra [8, Theorem 12.5].

We now list some facts about H_{mn} which we will later prove.

1. Let G_n denote the matrix group $\left\{ \begin{pmatrix} a & \mathbf{b} \\ 0 & I_n \end{pmatrix} : \mathbf{b} \in \mathbb{R}^n, a > 0 \right\}$. Note that G_1 is isomorphic as a Lie group to Aff^+. We prove in Theorem 2.1 that H_{mn} is isomorphic to $\mathbb{R}^{m-1} \times G_n$. Hence, after denoting the a coordinate in G_n by its natural logarithm, multiplication in H_{mn} becomes:

$$(v, a, \mathbf{b})(v', a', \mathbf{b}') = (v + v', a + a', \mathbf{b} + e^a \mathbf{b}') \tag{1.1}$$

for $v, v' \in \mathbb{R}^{m-1}, a, a' \in \mathbb{R}, \mathbf{b}, \mathbf{b}' \in \mathbb{R}^n$.

2. As shown in Lemma 2.3, the exponential map in H_{mn} is given by

$$\exp(v, a, \mathbf{b}) = \begin{cases} (v, a, \frac{\mathbf{b}}{a}(e^a - 1)) & \text{if } a \neq 0, \\ (v, 0, \mathbf{b}) & \text{if } a = 0. \end{cases}$$

where $(v, a, \mathbf{b}) \in \mathfrak{h}_{mn}$, the Lie algebra of H_{mn}. In particular exp is a bijection with its inverse analytic everywhere. Thus, H_{mn} is exponential.

We solve that Semigroup Problem for H_{mn} by proving two closely related conjectures. We first introduce some terminology.

Definition 1.1. A *maximal semigroup with nonempty interior* of a topological group G is a proper subsemigroup M of G with nonempty interior such that M is not a group and the only subsemigroups of G containing M are G and M. In this paper, the term *maximal semigroup* will always refer to those with nonempty interior.

Definition 1.2. A subset S of a topological group G is called *nonseparated* if it is not contained in a maximal semigroup.

Definition 1.3. A subset S of a group G is called *good* if the closure of the semigroup it generates is a group not contained in any connected codimension 1 subgroup.

Definition 1.4. A subset S of a group G is called *great* if it generates a dense semigroup. Additionally, for a fixed positive integer ℓ, we define $p \in G^\ell$ to be a *great ℓ-tuple* if the subset corresponding to p is great.

Definition 1.5. For groups G that satisfy [8, Theorem 12.5], we call the boundaries of the maximal semigroups *border subgroups*.

For H_{mn} exp is an analytic diffeomorphism. This implies that border subgroups are connected codimension 1 subgroups.

We now state the semigroup conjectures for H_{mn}.

Semigroup Conjecture 1: Let G be a topological group. Then, $S \subseteq G$ is good if and only if it is not contained in a maximal semigroup.

Semigroup Conjecture 2: Let G be a topological group. Then, for each positive integer ℓ, the set of great ℓ-tuples in G^ℓ is dense in the set of nonseparated ℓ-tuples in G^ℓ.

The first conjecture says that if $S \subseteq H_{mn}$ is not contained in a maximal semigroup, then the closed semigroup it generates is the same as the closed group it generates. The second conjecture says that for a typical such subset, the closed group it generates is H_{mn}. Together they imply the Semigroup Problem for H_{mn}

The rest of the paper is organized as follows. We first prove the facts about H_{mn} stated above and use them to show parts of the Semigroup

Conjecture 1. We then describe certain Lie group automorphisms of H_{mn}. Next, we prove an exact sequence lemma which is used to prove the conjectures inductively. We then prove properties of nonseparated semigroups and products in H_{mn} and use them to finish the proof of Semigroup Conjecture 1. Finally, we use a theorem of Kronecker and a structure lemma about good subsets in H_{mn} to prove Semigroup Conjecture 2.

2. Main Results

We first prove the structure theorem stated in the introduction.

Theorem 2.1. *The group H_{mn} is isomorphic as a Lie group to $\mathbb{R}^{m-1} \times G_n$.*

Proof. Any additive homomorphism from $\mathbb{R}^m$ to $\mathbb{R}^n$ is $\mathbb{Q}$-linear and hence $\mathbb{R}$-linear. Since ϕ is non-trivial, it must be surjective with $\mathbb{R}^m \cong \ker(\phi) \oplus H$. Let $K = \ker(\phi)$ and $h \in H$ nonzero. Suppose $\phi(h) = r$. Then, we define a map $\Phi : (K \oplus \mathbb{R}h) \ltimes \mathbb{R}^n \to \mathbb{R}^{m-1} \times G_n$ which maps $(v, xh, \mathbf{b}) \mapsto (v, \exp\left(\frac{x}{r}h\right), \mathbf{b})$. Φ is clearly bijective and smooth with smooth inverse. It is clear that Φ is an homomorphism. Hence, Φ is an isomorphism. $\square$

From now on we assume $H_{mn} = \mathbb{R}^{m-1} \times G_n$ with the product given by (1.1). It is clear from the above isomorphism that H_{mn} is solvable. Additionally, as it is diffeomorphic to $\mathbb{R}^{m+n}$, it is simply connected. Hence, H_{mn} satisfies the hypothesis of [8, Theorem 12.5]. Now, if H is a connected codimension 1 subgroup, then its Lie algebra $\mathfrak{h}$ is a hyperplane subalgebra. Thus, if S is contained in H, then S is contained in the maximal semigroups given by exponentiating each of the half-spaces determined by $\mathfrak{h}$. This gives the following theorem.

Theorem 2.2. *Let $S \subseteq H_{mn}$ be nonseparated. Then, S is not contained in any connected codimension 1 subgroup of H_{mn}.*

We now prove certain facts about the exponential map and the Lie group automorphisms of H_{mn}. These Lie group automorphisms will play an important role in the proof of the conjectures as the property of a subset being nonseparated, good or great are preserved by automorphisms.

Lemma 2.3. *The exponential map* $\exp : \mathfrak{h}_{mn} \to H_{mn}$ *is an analytic bijection with analytic inverse.*

Proof. Note that exp is always analytic. We show it is bijective. Since $H_{mn} \cong \mathbb{R}^{m-1} \times G_n$, it will suffice to show that exp is bijective with analytic inverse for G_n. Suppose $\mathfrak{g}_n$ is the Lie algebra for G_n. Then, $\mathfrak{g}_n$ consists of elements $(a, \mathbf{b})$, $a \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^n$, with multiplication $(a, \mathbf{b})(a', \mathbf{b}') = (aa', a\mathbf{b}')$. Thus $(a, \mathbf{b})^\ell = (a^\ell, a^{\ell-1}\mathbf{b})$. A calculation shows:

$$\exp(a, \mathbf{b}) = \begin{cases} (a, (e^a - 1)\frac{\mathbf{b}}{a}) & \text{if } a \neq 0, \\ (0, \mathbf{b}) & \text{if } a = 0, \end{cases}$$

and exp is clearly bijective. Consider now $\log = \exp^{-1}$,

$$\log(x, \mathbf{y}) = \begin{cases} (x, \frac{x}{e^x - 1}\mathbf{y}) & \text{if } x \neq 0, \\ (0, \mathbf{y}) & \text{if } x = 0. \end{cases}$$

which is clearly analytic for $x \neq 0$. For $x = 0$, the Taylor series of x and $e^x - 1$ have zero constant term and non-zero x term. Thus, the denominator is just a zero of order 1, and hence the fraction is analytic by standard complex analysis results. Thus, log is analytic everywhere. $\square$

Since exp is an analytic diffeomorphism, by [8, Theorem 12.5] the boundaries of maximal semigroups of H_{mn} are connected codimension 1 subgroups, images of hyperplane subalgebras of $\mathfrak{h}_{mn}$. Thus, for $M \subseteq H_{mn}$ maximal semigroup, if S is good, S is not contained in $Bd(M)$, the boundary of M. If S is separated, then S has an element x in the interior of some M. If U is the semigroup generated by S, then $x^{-1} \in \overline{U}$. Now, $\log(x), \log(x^{-1})$ are on opposite sides of $\log(Bd(M))$. Hence $\overline{U}$ is not contained in M, a contradiction as M is a closed semigroup. Thus S is nonseparated and a direction for Semigroup Conjecture 1 follows.

Theorem 2.4. *Let $S \subseteq H_{mn}$ be good. Then, S is nonseparated.*

We describe the (Lie group) automorphisms of $H_{mn} = \mathbb{R}^{m-1} \times G_n$. As H_{mn} is simply connected, one can consider the Lie algebra automorphisms. Let $\{X_1, \ldots, X_{m-1}\}$ be the canonical basis in $\mathbb{R}^{m-1}$, Y the canonical basis in $\mathbb{R}$ and $\{Z_1, \ldots, Z_n\}$ the canonical basis in $\mathbb{R}^n$. Then $\{X_1, \ldots, X_{m-1}, Y, Z_1, \ldots, Z_n\}$ is a basis for $\mathfrak{h}_{mn}$. The commutation relations are

$$[X_i, X_j] = [X_i, Y] = [X_i, Z_j] = [Z_i, Z_j] = 0, [Y, Z_i] = Z_i.$$

Thus, we immediately get the following automorphisms:

- Type A: Any automorphism ψ of $\mathbb{R}^{m-1}$ extends to an automorphism that is the identity on G_n.
- Type B: Any automorphism ψ of $\mathbb{R}^n$ extends to an automorphism which is the identity on the X_i and Y coordinates.
- Type C: The map that fixes X_i, Z_j and sends $Y \mapsto Y + \sum_i \alpha_i X_i + \sum_j \beta_j Z_j$ is a Lie algebra automorphism. Hence, for a given $(v, a, \mathbf{b}) \in H_{mn}$ there is a Lie group automorphism that sends it to $(v, a, \mathbf{0})$, defined by $(x, y, \mathbf{z}) \mapsto \left(x - \frac{y}{a}, y, \mathbf{z} - \frac{e^y - 1}{e^a - 1}\mathbf{b}\right)$.

We are ready to begin proving the Semigroup Conjecture 1 for H_{mn}. Our approach is by induction. We rely heavily on the following Lemma, which is proved in the setting of a general topological group. If G is a group and $X \subseteq G$, we denote by $U(X)$ the semigroup generated by X.

Lemma 2.5. *Consider an exact sequence of topological groups:*

$$0 \longrightarrow B \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0.$$

Let $S \subseteq G$ and let U be the semigroup generated by S. Then:

1. $\overline{\pi(\overline{U})} = \overline{\pi(U)}$.
2. *If $\overline{U}$ is a group, then so is $\overline{U(\pi(S))}$.*
3. *If S is great, then so is $\pi(S)$.*
4. *If $\overline{\pi(U)}$ is a group , $\pi(\overline{U})$ is closed and $\overline{U} \cap B$ is a group, then $\overline{U}$ is a group.*
5. *If $\pi(S)$ is great, $\pi(\overline{U})$ is closed and $\overline{U} \cap B$ is great, then S is great.*
6. *If S is nonseparated, then so is $\pi(S)$.*
7. *Let $B = \mathbb{R}^n$ and $G = \mathbb{R}^n \rtimes A$, with A second countable group, and suppose $\overline{U} \cap B$ is good. Then, $\pi(\overline{U})$ is closed.*

Proof.

1. Note that $\pi(U) = U(\pi(S))$. Clearly $\overline{\pi(\overline{U})} \supseteq \overline{\pi(U)}$. For the reverse, pick $x \in \overline{U} \backslash U$. Let $y = \pi(x)$. Let V be a neighborhood of y. Then, $\pi^{-1}(V)$ is a neighborhood of x, which hence contains an element of U. Thus, V contains an element of $\pi(U)$. Hence, $\pi(\overline{U}) \subseteq \overline{\pi(U)}$ and thus $\overline{(\pi(\overline{U}))} = \overline{\pi(U)}$.
2. Note that $\overline{U(\pi(S))} = \overline{\pi(U)} = \overline{\pi(\overline{U})}$, which is a group if $\overline{U}$ is a group.
3. If $\overline{U} = G$, then $\overline{U(\pi(S))} = A$, as it contains $\pi(\overline{U})$.
4. If $\pi(\overline{U})$ is closed, then it is equal to $\overline{\pi(U)}$, which is a group. Now, to show $\overline{U}$ is a group, since closures of semigroups are semigroups, it suffices to show that an inverse exists in $\overline{U}$ for every element in $\overline{U}$. Pick $x \in \overline{U}$. Then, there exists an inverse in $\pi(\overline{U})$ for $\pi(x)$. Thus, there exists $y \in \overline{U}$ such that $yx \in \overline{U} \cap B$. Thus, $x^{-1}y^{-1} \in \overline{U} \cap B$. Thus, $x^{-1} = x^{-1}y^{-1}y \in \overline{U}$.
5. Since $\pi(\overline{U})$ is closed, and $\pi(S)$ is great, $\pi(\overline{U})$ is great. Additionally, as $\overline{U} \cap B$ is great, it must be all of B, as it is a closed semigroup. Thus, $\overline{U}$ contains B and a representative for each coset in A. Thus, $\overline{U} = G$.
6. Let M be a maximal semigroup of A. Then, by [8, Lemma 3.12], $\pi^{-1}(M)$ is a maximal semigroup of G and it obviously has nonempty interior by continuity of π. Thus, as S is nonseparated,

S is not contained in $\pi^{-1}(M)$. Thus, $\pi(S)$ is not contained in M. Hence, $\pi(S)$ is nonseparated.

(7) Let y be a limit point of $\pi(\overline{U})$. Choose a sequence $\{(x_i, y_i)\} \subset \overline{U}$ such that $y_i \to y$. As $\overline{U} \cap B$ is good, it is not contained in any hyperplane. Thus there are n linearly independent vectors in it, together with their opposites. As left multiplication by elements of B does not change y_i, by suitable multiplying with the basis elements or their opposites, one finds a new sequence $\{(x_i', y_i)\}$ in $\overline{U}$ such that $\{x_i'\}$ is bounded. Choosing a convergent subsequence gives $(x, y) \in \overline{U}$. Thus $\pi(\overline{U})$ is closed.

□

Lemma 2.6. *Identifying G_1 with its Lie algebra $\mathfrak{g}_1 = \mathbb{R}^2$, the border subgroups are the (boundary) curves $y = l(e^x - 1), (x, y) \in \mathfrak{g}_1$, where $l \in \mathbb{R}$, and $x = 0$. We call l the slope of the boundary curve. Moreover,*

(1) *Every nonzero point in $\mathbb{R}^2$ belongs to a unique boundary curve.*
(2) *For $l \geq 0$, the boundary curve is contained in the first and third quadrants. For $l \leq 0$, the boundary curve is contained in the second and fourth quadrants.*
(3) *Let $l \leq 0$. For points z in the fourth quadrant, if l_z is the slope of the boundary curve through z, then $l_z < l$ if and only if z is below the boundary curve of slope l. For z in the second quadrant, z is below the boundary curve of slope l if and only if $l_z > l$.*

Proof. A codimension 1 subspace of $\mathfrak{g}_1$ is of dimension 1, so it is a subalgebra. The border subgroups are images under exp of these subalgebras. If the subalgebra is $\{t(a, b) : t \in \mathbb{R}\}$, then by Lemma 2.3, the associated boundary curve is given by $\{(ta, \frac{b}{a}e^{ta} - 1) : t \in \mathbb{R}\}$ which is the curve $y = l(e^x - 1), l \in \mathbb{R}$, or $x = 0$. Now, (1) and (2) follows from the definition of the curves. For (3), note that for $z = (x_0, y_0)$, $l_z = \frac{y}{e^{x_0} - 1}$, and hence for z in the fourth quadrant (resp. second quadrant), z is below the curve $y = l(e^x - 1)$ if and only if $l > l_z$ $(l < l_z)$ as $e^{x_0} - 1 > 0$ (< 0). □

Lemma 2.7. *Let $S \subseteq H_{mn}$ nonseparated, $U = U(S)$ and $z_0 = (w, a, \mathbf{b})$, $z = (w', a', \mathbf{b}') \in \overline{U}$ with $a < 0, a' > 0$. Then $\left(0, 0, \frac{\mathbf{b}}{1-e^a} + \frac{\mathbf{b}'}{e^{a'}-1}\right) \in \overline{U}$.*

Proof. Let $v_0 = (w, a), v = (w', a')$. Let π be projection of $H_{mn} \cong \mathbb{R}^m \ltimes \mathbb{R}^n$ onto $\mathbb{R}^m = B$. As S is nonseparated, by Lemma 2.5, $\pi(S)$ is nonseparated. By [1, Lemma 2.12], there exists a finite subset $F = \{v_1, \ldots, v_l\}$ of $\pi(S)$ such that its convex hull contains 0. Denote $v = v_{l+1}, z = z_{l+1}$. Thus, for small $\alpha_0, \alpha_{l+1} > 0$, $-\alpha_0 v_0 - \alpha_{l+1} v_{l+1}$ is in the convex hull of F and

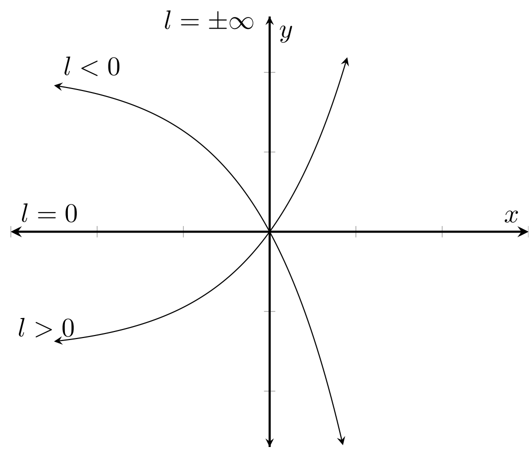

FIGURE 1. Different types of boundary curves

there exist $\alpha_i \geq 0, 1 \leq i \leq l$, such that $\sum_{i=0}^{l+1} \alpha_i v_i = 0$. Without loss, assume $\alpha_i > 0$. Thus $\sum_{i=0}^{l+1} t\alpha_i v_i = 0$ for t positive integer. Denote by $\hat{t}$ the projection of $(t\alpha_0, \ldots, t\alpha_{l+1})$ in the l-torus $\mathbb{T}^l$. As $\mathbb{T}^l$ is compact, a subsequence t_k exists such that $\hat{t_k} \to 0$. Denote by $t_i^{(k)}$ the integer closest to $t_k\alpha_i$. Then, as $t_k\alpha_i$ is increasing and unbounded, $t_i^{(k)} \to \infty$. Now, as $t_k\alpha_i$ converges to 0 in $\mathbb{T}$, we note that for each $\epsilon > 0$, there exists an $k_0 > 0$ such that for $k > k_0$, there exists some integer r such that $|r - t_k\alpha_i| < \epsilon$. But then, $|t_i^{(k)} - t_k\alpha_i| < \epsilon$, thus $t_i^{(k)} - t_k\alpha_i \to 0$ and $\sum_{i=0}^{l+1} t_i^{(k)} v_i \to 0$.

We now assume without loss that $v_1, \ldots, v_l$ are in nondecreasing order of their a_i coordinate. Since $v_i \in \pi(S)$ there exists $z_i = (v_i, b_i) \in S$. By induction $z_i^t = (tv_i, b\frac{1-e^{ta_i}}{1-e^{a_i}})$ with the fraction equal to t for $a_i = 0$. Thus,

$$z_0^{t_0^{(k)}} \cdots z_{l+1}^{t_{l+1}^{(k)}} = \left(\sum_i t_i^{(k)} v_i, \sum_i \frac{1 - e^{t_i^{(k)} a_i}}{1 - e^{a_i}} e^{\sum_{j=0}^{i-1} t_j^{(k)} a_j} \mathbf{b_i} \right).$$

Now, for $i \neq l+1$, $\sum_{j=0}^{i} t_j^{(k)} a_j \to -\infty$. This is clear when $a_i \leq 0$ and holds for $a_i > 0$ as otherwise, the sum for $i = l+1$ would converge to ∞ and not 0 as constructed. Thus, for $i \neq 0, l+1$, $a_i \neq 0$,

$$\frac{1-e^{t_i^{(k)}a_i}}{1-e^{a_i}}e^{\sum_{j=0}^{i-1} t_j^{(k)}a_j} = \frac{e^{\sum_{j=0}^{i-1} t_j^{(k)}a_j} - e^{\sum_{j=0}^{i} t_j^{(k)}a_j}}{1-e^{a_i}} \to 0$$

and for $i \neq 0, l+1, a_i = 0$, as exponentials dominate over a linear term, we get the same result. Hence,

$$z_0^{t_0^{(k)}} \cdots z_{l+1}^{t_{l+1}^{(k)}} \to \left(0, \frac{\mathbf{b_0}}{1-e^{a_0}} + \frac{\mathbf{b_{l+1}}}{e^{a_{l+1}}-1}\right) = \left(0, \frac{\mathbf{b}}{1-e^{a}} + \frac{\mathbf{b'}}{e^{a'}-1}\right) \in \overline{U}.$$

□

Lemma 2.8. *Let $S \subseteq G_1$ nonseparated and $U = U(S)$. Then U contains up to automorphism elements $(a,0),(c,d)$ $a,d > 0, c < 0$. In fact, up to an arbitrary small perturbation, S contains such elements.*

Proof. As $x \leq 0$ is a maximal semigroup, S contains (a,b), $a > 0$. By a type C automorphism, assume $b = 0$. Additionally, as $x \geq 0$ is a maximal semigroup, S contains (c,d) with $c < 0$. By a type B automorphism, we may assume $b \geq 0$. If $b > 0$, which may be done by an arbitrary small perturbation, we are done and S will contain such elements. If $b = 0$, then choose (c',d') with $d' > 0$, which exists in S as $y \leq 0$ is a maximal semigroup. Then, taking suitable multiples of $(c,0)(c',d')$ gives us the desired element in U. □

Lemma 2.9. *Suppose a semigroup U contains elements z, z' respectively in the interior the second and fourth quadrant, such that $l_z > l_{z'}$. Then, U contains an element in the interior of the third quadrant.*

Proof. It suffices to show $\overline{U}$ contains such an element. Let $z = (\ln a, b)$, $z' = (\ln a', b')$. Then, by the hypotheses, $\frac{b}{a-1} > \frac{b'}{a'-1}$. By Lemma 2.7, $z_0 = \left(0, \frac{b}{1-a} + \frac{b'}{a'-1}\right) \in \overline{U}$. As the second coordinate is less than 0, there exists k positive integer such that $k\left(\frac{b}{1-a} + \frac{b'}{a'-1}\right) < -b$. Then, $z_0^k z$ is an element in $\overline{U}$ with the desired property. □

Lemma 2.10. *Suppose $S \subseteq G_1$ is nonseparated with $U = U(S)$ containing $(a,0),(a',b')$, $a,b' > 0, a < 0$. Then, U contains (c,d), $c,d < 0$.*

Proof. Suppose S contains z, z_0 two such elements. As $y \geq 0$ is a maximal semigroup of G_1, S contains an element $z' = (c,d)$ with $d < 0$. If $c < 0$ we are done. If $c = 0$, then

$$z_0^l z' = \left(la', e^{la'}d + b'\frac{1-e^{la'}}{1-a'}\right).$$

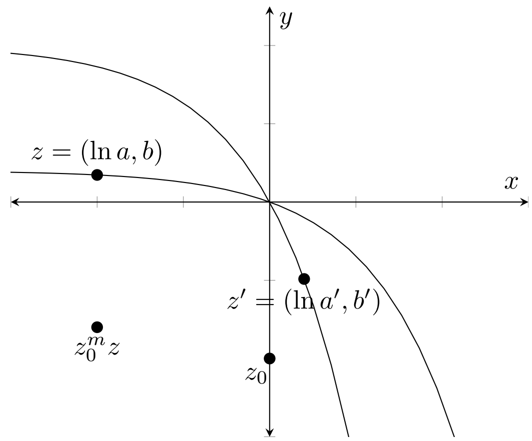

FIGURE 2. Obtaining a point in the third quadrant

As $l \to \infty$, $e^{la'} \to 0$. Hence for large l, this element is in the interior of the third quadrant. So we may assume $c > 0$ and hence that S contains elements in the interiors of the second quadrant and of the fourth quadrant. Suppose, by contradiction, that U does not contain an element in the interior of the third quadrant. Define

$$L := \{l_z : z \in \text{Interior of Second Quadrant}\},$$
$$L' := \{l_z : z \in \text{Interior of Fourth Quadrant}\}.$$

By the argument above, $L, L' \neq \emptyset$ and also, by Lemma 2.6, $L, L' \subseteq (-\infty, 0)$. By Lemma 2.9, there are no $l \in L, l' \in L'$ such that $l > l'$. Thus, as L' is nonempty, L is bounded above, L' is clearly bounded below and $\sup L \leq \inf L' = l$. Thus, for each point z in the interior of the second quadrant, $l_z \leq l$ and hence z is in the maximal semigroup above l and for z in the interior of the fourth quadrant $l_z \geq l$ giving the same result.

As S is nonseparated, there exists a point in S below the boundary curve of slope l. As this cannot be in the interior of the second, third or fourth quadrants, and $l < 0$, this point must be on either the negative x or y-axis. In either case, we repeat a similar argument to the $c = 0$ case to get a point in the interior of the third quadrant, a contradiction. □

As a final intermediate step, we prove the following Lemma regarding good and nonseparated subsets in $\mathbb{R}^n$. Note that these notions are equivalent for $\mathbb{R}^n$ by [1, Lemma 2.12].

Lemma 2.11. *Let $S \subseteq \mathbb{R}^n$. If for each $X \subseteq \mathbb{R}^n$ 1-dimensional subspace and $\tau : \mathbb{R}^n \to X$ projection, $\tau(S)$ is nonseparated, then S is nonseparated.*

Proof. Suppose, for contradiction, that S is contained in some closed half-space M. Let H be the boundary hyperplane. Choose a complementary one dimensional subspace X and let τ be projection onto X with kernel H. Then, without loss of generality, $\tau(S) \subseteq [0, \infty)$ as S is contained on one side of H, a contradiction. □

We now prove the following theorem, which along with Theorems 2.2 and 2.4 proves the Semigroup Conjecture 1 for H_{mn}.

Theorem 2.12. *Let $S \subseteq H_{mn}$ nonseparated, $U = U(S)$. Then $\overline{U}$ is group.*

Proof. Note that H_{mn} is the central term in a short exact sequence with image $\mathbb{R}^m = C \times A = \mathbb{R}^{m-1} \times \mathbb{R}$ and kernel $\mathbb{R}^n = B$. Thus, by Lemma 2.5, (4) and (7), it suffices to show that $\overline{U} \cap B$ is good. We use the previous lemma. Call $\overline{U} \cap B = S'$. Applying a type C automorphism, S contains $z = (0, a, 0)$ for some $a > 0$. Choose an arbitrary 1-dimensional subspace $X \subseteq B$ and let τ be a projection. We use an automorphism to let X be the B_1 coordinate and $B_2, \ldots, B_n$ be in the kernel of τ. We will show that $\tau(S')$ is nonseparated. Since this holds for arbitrary X, τ, we are done.

Let π be projection onto the A coordinate and the B_1 coordinate. This is a surjective group homomorphism onto G_1. By 2.5, $\pi(S)$ is nonseparated. Hence, as $\pi(U) = U(\pi(S))$, we may assume U contains, by the same method as Lemma 2.8 an element $z' = (x'', a', \mathbf{b}')$ with $a' < 0, b_1' > 0$, as the automorphism that may be required simply flips the basis vector for X to its negative. Thus, using Lemma 2.10 on $\pi(U)$, we see that U contains an element $z'' = (x'', a'', \mathbf{b}'')$ with $a'', b_1'' < 0$. Thus, applying Lemma 2.7, S' contains $(0, 0, \frac{\mathbf{b}'}{1-e^{a'}})$ and $(0, 0, \frac{\mathbf{b}''}{1-e^{a''}})$. Thus, $\tau(S')$ contains positive and negative elements and is hence nonseparated. □

We will now focus on proving the Semigroup Conjecture 2 for H_{mn}. Note that the notions of good and nonseparated are equivalently for H_{mn}.

Lemma 2.13. *Let $S \subseteq H_{mn}$ be good. Then, there exists an automorphism Φ of H_{mn}, such that $(x, \ln a, \mathbf{0}), (x_i, \ln c_i, |1 - c_i|\mathbf{e_i}) \in \Phi(S)$ where $\ln a > 0$ and $\mathbf{e_i}$ is the i-th standard basis vector for $B \cong \mathbb{R}^n$. Additionally, we can choose Φ such that either $(x', \ln a', 0) \in \Phi(S)$ for $a' < 1$ or $c_1 < 1$.*

Proof. We get $(x, \ln a, \mathbf{0}) \in S$ by a type C automorphism. It now suffices to prove the following statement for S by induction on i: for each i from 1 to n, there exists an automorphism Φ_i such that $(x_j, \ln c_j |1 - c_j|\mathbf{e_j}) \in \Phi_i(S)$ for j from 1 to i.

Base Case $i = 1$. By separation, there exist $(x_1, \ln c_1, \mathbf{v})$ with $\mathbf{v} \neq 0$, which we can then send by type B automorphism to $(x_1, \ln c_1, |1 - c_1|\mathbf{e_1})$.

Additionally, if there exists any such element with $c_1 < 1$ we can choose $c_1 < 1$. otherwise, by separation, there exist $(x', \ln a', \mathbf{0})$ for $a' < 1$.

Inductive Step: Suppose the statement holds for $i-1$ with $i \leq n$. By separation, there exists an element $(x_i, \ln c_i, |1-c_i|\mathbf{v})$ with $v_i \neq 0$. Then, as $|1-c_1|\mathbf{e_1}, \ldots, |1-c_{i-1}|\mathbf{e_{i-1}}, \mathbf{v}$ are linearly independent, we can find a type B automorphism to get S into the desired form. $\square$

Lemma 2.14. *If $S \subseteq \mathbb{R}^n$ contains the standard basis $\mathbf{e_i}$ and an element $\mathbf{v}$ with $v_i < 0$ and $\{1, v_1, \ldots, v_n\}$ $\mathbb{Z}$-linearly independent, then S is great.*

Proof. Note that the convex hull of S contains 0 in the interior. Thus, S is good by [1, Lemma 2.12]. Thus, the closure of the semigroup generated by S is the closed group generated by S and hence it suffices to prove that the group generated by S is dense. Now, as S contains the standard basis, $\overline{U}$ contains $\mathbb{Z}^n$. Hence, it suffices to show $\pi(\overline{U})$ is dense, where π is the projection onto $\mathbb{T}^n$. As $\{1, v_1, \ldots, v_n\}$ are $\mathbb{Z}$ linearly independent, there does not exist any nontrivial relation $c_0 + c_1 v_1 + \cdots + c_n v_n = 0$. Thus, by a result of Kronecker cited in [9], $\mathbb{Z}\mathbf{v} \mod 1$ is dense in $\mathbb{T}^n$. $\square$

Lemma 2.15. *Fix nonzero $\alpha \in \mathbb{R}$. The set $X := \{\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n : \{\alpha, v_1, \ldots, v_n\}$ is $\mathbb{Z}$ linearly dependent$\}$ has measure zero in $\mathbb{R}^n$.*

Proof. Let $\phi : \mathbb{R}^n \to \mathbb{R}, \phi(x_1, \ldots, x_n) = c_0 + c_1 x_1 + \cdots + c_n x_n, c_i \in \mathbb{Z}, i > 0$, and $c_0 \in \mathbb{Z}\alpha$ be a nonzero function. If ϕ is constant, then $\phi^{-1}(0)$ is empty and is hence measure zero. If ϕ is nonconstant, ϕ is a submersion as $d\phi$ is a nonzero constant. Thus, $\phi^{-1}(0)$ is a submanifold of codimension 1, and hence has measure zero. Now, X is the union of $\phi^{-1}(0)$ over all such ϕ. As there are only countable many such ϕ, X has measure zero. $\square$

Lemma 2.16. *For any ℓ, the set of great ℓ-tuples of $\mathbb{R}^n$ is dense in the set of nonseparated (i.e good) ℓ-tuples of $\mathbb{R}^n$.*

Proof. As nonseparation is equivalent to good in $\mathbb{R}^n$, we will prove denseness in good ℓ-tuples. Fix ℓ. If there are no good subsets of size ℓ, the lemma is vacuously true. Suppose S is a subset corresponding to a good ℓ-tuple. We need to find a great S' arbitrarily close to S. After an automorphism, S contains the standard basis. Additionally, as it is good, and $U = U(S)$ contains an element $\mathbf{w}$ with $w_i > 0$, $\overline{U}$ and hence U contains an element $\mathbf{v}$ with $v_i < 0$. Suppose $\mathbf{v} = \sum_{i=1}^k \mathbf{b_i}$ with $\mathbf{b_i} \in S$. Now, we may assume $\mathbf{b_1}$ is not a standard basis vector and that $\mathbf{b_1}, \ldots, \mathbf{b_j}$ are equal.

Pulling back a neighborhood V of $\mathbf{v}$ under the map $\alpha(x) = jx + \sum_{i=j+1}^k \mathbf{b_i}$ gives a neighborhood of $\mathbf{b_1}$. By Lemma 2.15, we can find in V an element $\mathbf{v}'$ such that each $v'_i < 0$ and $\{1, v'_1, \ldots, v'_n\}$ $\mathbb{Z}$ are linearly independent. Then, if $\alpha(\mathbf{b}') = \mathbf{v}'$, replacing the subset S with S' by changing $\mathbf{b_1}$ to $\mathbf{b}'$, which can be done by an arbitrarily small perturbation, we get a great subset by Lemma 2.14. $\square$

We now prove Semigroup Conjecture 2 for H_{mn}.

Theorem 2.17. *For any ℓ, the set of great ℓ-tuples of $H_{mn} = \mathbb{R}^m \ltimes \mathbb{R}^n = \mathbb{R}^{m-1} \times G_n$ is dense in the set of nonseparated ℓ-tuples of H_{mn}.*

Proof. If there are no good subsets of size ℓ, the theorem is true. Suppose S is a good subset of size ℓ. We will apply to S an arbitrary small perturbation to get a great subset. Let π_1 be projection onto $\mathbb{R}^m = C \times A = \mathbb{R}^{m-1} \times \mathbb{R}$ and let π be projection onto G_n. Applying a type C automorphism, we assume S contains $(0, a, \mathbf{0})$ for $a > 0$. Additionally, as $\pi(S)$ is good, by Lemma 2.13, we may assume S contains $(w_i, \ln c_i, |1 - c_i|\mathbf{e_i})$ where either $c_1 < 1$ or S also contains $(w', \ln a', \mathbf{0}), a' < 1$.

Now, as $\pi_1(S)$ is good, we may, by Lemma 2.16 perturb S slightly, leaving the $B = \mathbb{R}^n$ coordinates unchanged, to get a subset S' such that $\pi_1(S')$ is great. As the consecutive perturbations can still be made arbitrarily small, we may assume $S = S'$ by forgetting that S is good. However, S still contains $z = (w, \ln a, \mathbf{0}), \ln a > 0, z_i = (w_i, \ln c_i, |1 - c_i|\mathbf{e_i})$ with either $c_1 < 1$ or an element $z' = (w', \ln a', \mathbf{0})$ with $a' < 1$ as signs can be left unchanged. Additionally, note that $U = U(S)$ originally contained an element $z'' = (w'', \ln c, \mathbf{b})$ with each $b_i < 0, \ln c < 0$ and $b_{i+1} < b_i - 2(1-c)$. As this is a finite sum of elements from S and the perturbation is arbitrarily small, U still contains such an element, even after perturbation. Suppose $z'' = \prod_{i=1}^{k} y_i$. Now at least one of the y_i is not z, z', z_i. Let $\{y_{j_1}, \ldots, y_{j_r}\}$ be the list of all such elements. Consider the continuous map $\alpha : H_{mn}^r \to H_{mn}$ defined by

$$\alpha(x_1, \ldots, x_r) = y_1 \cdots y_{j_1 - 1} x_1 y_{j_1 + 1} \cdots x_r y_{j_r + 1} \cdots y_k.$$

The preimage of a neighborhood of z'' is a neighborhood of $(y_{j_1}, \ldots, y_{j_r})$. Thus, using Lemma 2.15, by an arbitrarily small perturbation of each of these, and hence of S, we may change z'' to $z'' = (w'', \ln c, \mathbf{b})$ such that $\frac{\ln c}{\ln a} \notin \mathbb{Q}$, $\{1 - c, b_1, \ldots, b_n\}$ are $\mathbb{Z}$-linearly independent and $b_{i+1} < b_i - 2(1-c)$. Now, $\pi_1(S)$ is great. So by Lemma 2.5, it suffices to prove $\overline{U} \cap B$ is great. We distinguish two cases.

Case 1: Suppose $(w, \ln a', \mathbf{0}) \in S, a' < 1$. By Lemma 2.7 with z', z_i for $c_i > 1$, and with z, z_i for $c_i < 1$, and then again by Lemma 2.7 with z'', z, one has $(0, 0, \mathbf{e_i}), \left(0, 0, \frac{\mathbf{b}}{1-c}\right) \in \overline{U} \cap B$. As $\left\{1, \frac{b_1}{1-c}, \ldots, \frac{b_n}{1-c}\right\}$ is $\mathbb{Z}$-linearly independent, by Lemma 2.14, $\overline{U} \cap B$ is good.

Case 2: Suppose S contains the element $(w_1, \ln c_1, |1 - c_1|\mathbf{e_1})$ with $c_1 < 1$. Then, using Lemma 2.7 $\overline{U} \cap B$ contains $(0, 0, \mathbf{e_1}), (0, 0, \frac{\mathbf{b}}{1-c})$ and either $(0, 0, \mathbf{e_i})$ or $(0, 0, \mathbf{e_i} + \mathbf{e_1})$ for each i bigger than 1. We now apply an automorphism of type C which takes $(0,0,\mathbf{e_i}+\mathbf{e_1})$ to $(0,0,\mathbf{e_i})$ where necessary to get the standard basis of $\overline{U}\cap B$. The matrix of this transformation is the identity with some extra -1 entries in the first row. Hence, as $b_{i+1} < b_i - 2(1-c)$, the transformation still gives an element $(0,0,\mathbf{b})$ with each $b_i < 0$. Additionally, $\{1, b_1, \ldots, b_n\}$ is still $\mathbb{Z}$-linearly independent, as the transformation merely sends some b_i to $b_i - b_1$. Thus, applying Lemma 2.14 shows that $\overline{U}\cap B$ is great. □

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