Tiling Deficient Boards Using L-Pentominoes

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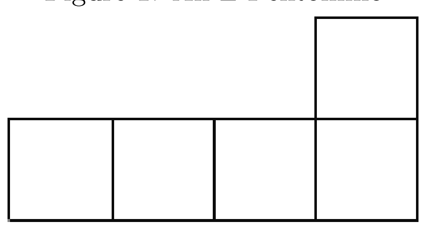
Abstract

Here we show which deficient boards can be tiled by L-shaped pentominoes. In a later paper, we will show which deficient rectangles can be tiled by the same shapes.

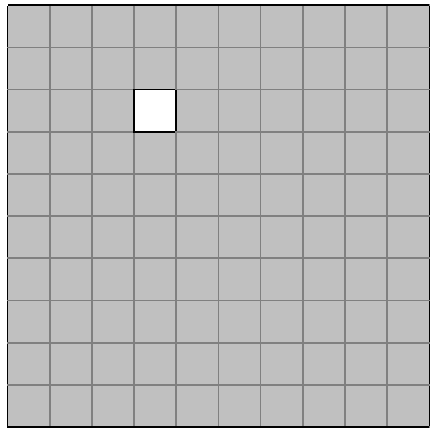
A pentomino is a type of polyomino consisting of five 1×1 squares. This group of objects was introduced by Solomon P. Golomb in the mid-20th century, and it has been extremely popular in the field of combinatorics. A kth order polyomino is a 2-D figure consisting of k unit squares joined at the edges. There is one type of 2nd order polyomino, namely the domino, two types of 3rd order polyominoes that make up the tromino class, five types of 4th order polyminoes that make up the tetromino class, 12 types of 5th order polyominoes that make up the pentomino class, and so on. We are working with a particular type of pentomino called an L pentomino. An L-pentomino is a tile containing five 1×1 squares, where four of these squares are in a line and the last one is attached to one end. It is shown in Figure 1. We will be considering all eight symmetries of this L pentomino.

Figure 1: An L-Pentomino



In this paper, we find which deficient boards can be tiled using all eight symmetries of this L pentomino. A board is an $n \times n$ square that is broken up into 1×1 tiles. It is easy to see that, any board where n>5 and the area is divisible by 5 can be completely tiled by using L pentominoes, because we can make 2×5 and 5×2 blocks using the pentominoes. This is an accepted result. However, we are interested in whether the board is tileable after we take out a single 1×1 square from it. This board is now called a *deficient* board. An example of a deficient board is shown below.

Figure 2: A Deficient Board



A lot of the proofs that follow rely on a powerful tiling program called Polysolver. If the type of tiles desired and the dimensions of the board are specified, the program will tell you whether that board can be tiled or not, and further will give you all of the possible tilings of the board. A lot of our proof rests on this program's findings. We will frequently say "using Polysolver...", indicating that we checked the case or cases on Polysolver.

To record which 1×1 squares we are taking out, we label each unit square on the board by the coordinates of its upper right corner if we were to put the board in the 1st quadrant, with the lower left corner at the origin. For example, a unit square placed at the origin would have coordinates (1,1) because its upper right corner has x-coordinate 1 and y-coordinate 1.

Since the L pentomino, as shown in Figure 1, has five 1×1 squares, it follows that the area of the board covered by the tiles must be divisible by 5. Since we are taking out a 1×1 square, the total area of the board must be $1, 6 \pmod{10}$. This condition alone isn't enough to tell us which deficient boards can or cannot be tiled. The theorems that follow are the conditions that must be satisfied for an $n \times n$ board to be tiled.

Theorem 1. *All deficient $n \times n$ boards are tileable if and only if:*

1. $n = 1, 4, 6, 9 \pmod{10}$
2. $n \geq 14$

We have four cases to consider here: the $n = 1 \pmod{10}$ case, the $n = 4 \pmod{10}$ case, the $n = 6 \pmod{10}$ case, and the $n = 9 \pmod{10}$ case. Let's consider the $n = 4 \pmod{10}$ case first.

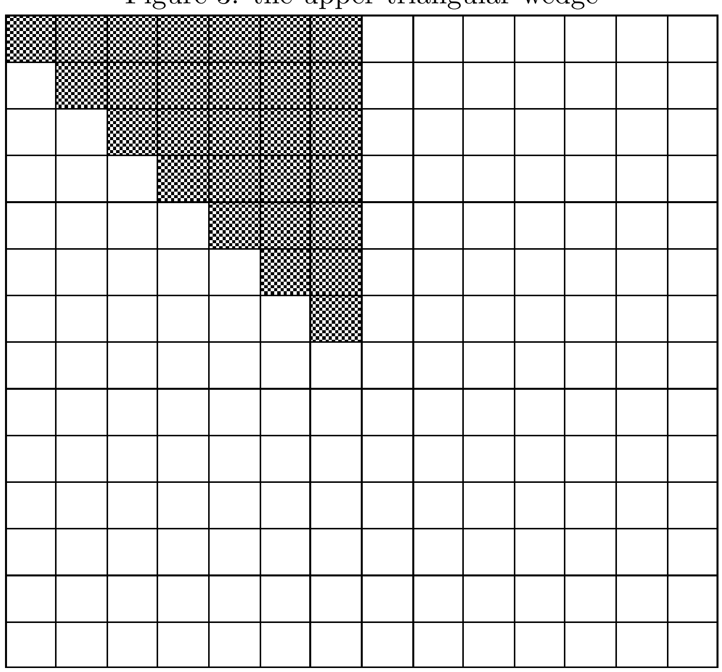
Case 1. $n = 4 \pmod{10}$

Proof. Clearly, a deficient 4×4 board is too small to be tiled using L pentominoes. There is only space for three pentominoes, and no matter what combination is used, there is no way to tile a board this small. So we must jump up to the next case: namely, a 14×14 board. To show that a 14×14 board can be completely tiled, we introduce a lemma.

Lemma 1. *To show that any square can be taken out of a board and the board can still be tiled, we only need to show that any tile can be taken out of a $n/2$ by $n/2$ triangular wedge that shares an edge with the board and the board can still be tiled.*

We are using all eight types of L pentominoes, because we allow for reflections and rotations. In addition, a square has eight symmetries. So if we know what happens when we take a square out of this triangular wedge, we by extension know what would happen if we took out a square in seven other places. Unless of course that square was the edge, in which case we would know what would happen if we took a square out of the other three edges. Figure 3 shows one example of a triangular wedge that, if the behavior of every situation where a tile is taken out of that wedge is known, the rest of the board's behavior can be known as well. Figure 3 also shows the triangular wedge that we use throughout this paper to support our proofs.

Figure 3: the upper triangular wedge

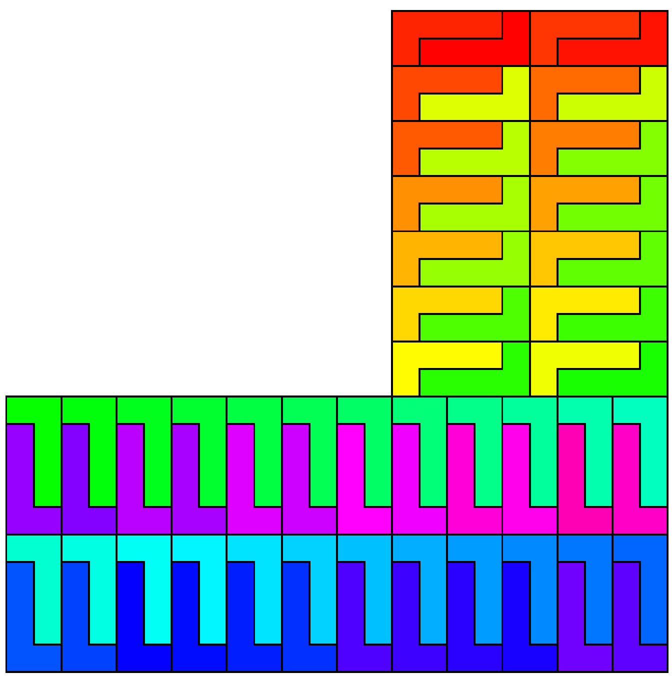


Now that we only need to test those squares that are in the triangular wedge, all that is left is to plug each case into Polysolver. After doing so, Polysolver concludes that every deficient 14×14 board can be tiled using L pentominoes. However, this is only one of an infinite amount of $4 \times 4 \pmod{10}$ boards. We also need to prove that the rest of these boards can be tiled. To do so, we introduce another lemma.

Lemma 2. *If any deficient board with side length n can be tiled, all $m \times m$ boards such that $m \pmod{10} = n \pmod{10}$ and $m > n$ can be tiled as well.*

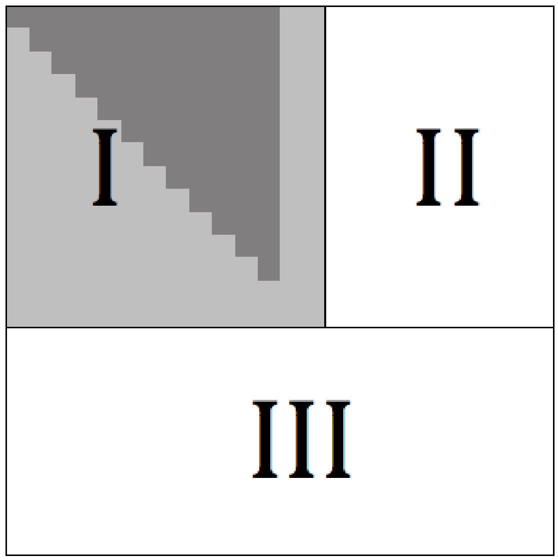
Let's take a look at the 24×24 board. Recall we only need to show that any unit square can be taken out of the wedge shown in the figure above and the board can still be tiled to prove that every deficient 24×24 board can be tiled. The wedge is 12 units long and 12 units wide. We know already that all deficient 14×14 boards can be tiled. Let's place the 14×14 board *inside* the 24×24 board. This board completely covers all of the squares inside the wedge. As stated before, we know that every 14×14 deficient board can be tiled, so by extension, the wedge lemma is satisfied, provided that the L shape that lies outside of the 14×14 board but inside the 24×24 board can be tiled. This tiling is indeed possible, regardless of whether n is odd or even. The following figure shows one of the simpler tilings of this L shaped region, using the 2×5 and 5×2 blocks that we can make by putting two complementary L pentominoes together.

Figure 4: Tiling of L shaped region left by taking 14×14 board out of 24×24 board



To finish proving this lemma, we need to prove that when we take out an $n \times n$ board from an $m \times m$ board and m and n satisfy the aforementioned conditions, the L shaped region left behind is always tileable. To show this, we divide the board into three regions, as shown in the next figure.

Figure 5: Board divided into three regions



Region I is the $n \times n$ board contained inside the $m \times m$ board. We need to prove that regions II and III can be tiled in all cases. Let's start with Region II. The dimensions of the Region II rectangle are $10 \times n$. Because of the 10, this region is easily tileable, because if n is even, you can simply tile the region using 2×5 rectangles all the way down. If n is odd, start with five 5×2 rectangles. At that point, you will have a $10 \times n$=5 rectangle, which can then be tiled using 2×5 rectangles.

Region III is tiled much the same way. Its dimensions are $10 \times m$. If m is even, the whole rectangle can be tiled using 5×2 rectangles, and if m is odd, start with five 2×5 rectangles in order to get the dimension to be even, and then proceed to tile the rest of the region using 5×2 rectangles.

$\square$

Case 2. $n = 6 \pmod{10}$

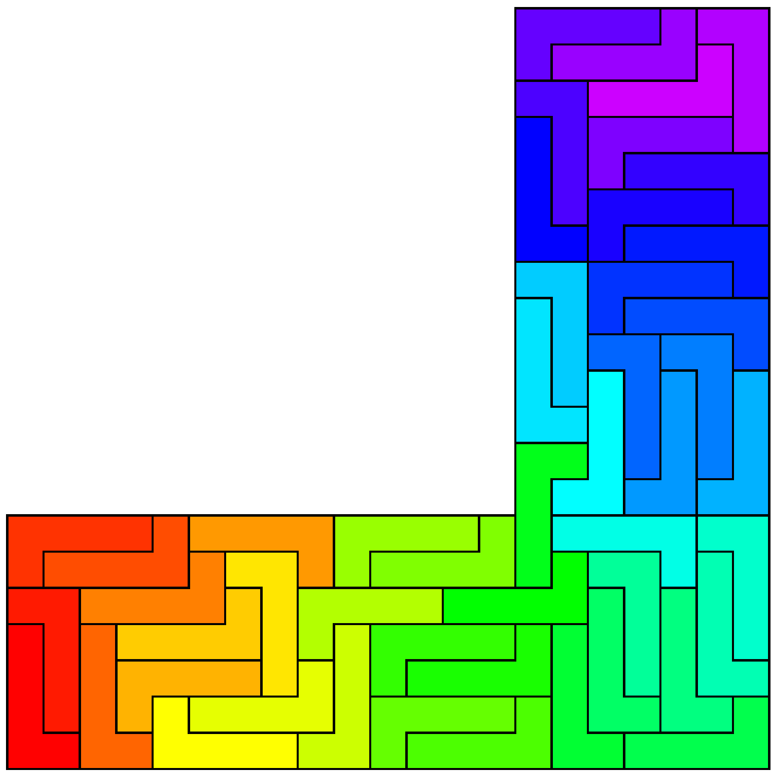
Using Polysolver, we conclude that all 16×16 deficient boards can be tiled. Using the above lemmas, we conclude by extension that all $6 \pmod{10}$ deficient boards can be tiled, provided that $n \geq 16$.

Case 3. $n = 9 \pmod{10}$

Using Polysolver, we conclude that all 19×19 deficient boards can be tiled. Using the above lemmas, we conclude by extension that all $9 \pmod{10}$ deficient boards can be tiled, provided that $n \geq 19$.

Case 4. $n = 1 \pmod{10}$

Figure 6: A tiling of the L shaped region created by taking a 14×14 board out of a 21×21 board



With the smallest case satisfying the two conditions, namely $n = 21$, we could not simply put the 11×11 board into the 21×21 board, because not all deficient 11×11 boards can be tiled. Instead of plugging each case into Polysolver, however, we can insert the 14×14 board into the 21×21 board and check whether the L shape left behind can be tiled. It indeed can, as shown in the figure above. By extension, all deficient 21×21 boards can be tiled. Again, by extension, all deficient $n \times n$ boards such that $n = 1 \pmod{10}$, and $n \geq 21$ can be tiled.

Theorem 2. *If $n \leq 14$, some restrictions apply:*

1. *If $n = 6$, then the deficient board is tileable if and only if the deficient square has coordinates $(1,2)$, $(2,1)$, $(5,1)$, $(6,2)$, $(1,5)$, $(2,6)$, $(5,6)$ or $(6,5)$*

2. *If $n = 9$, then the deficient board is tileable if and only if the deficient square has coordinates $(2,3)$, $(2,7)$, $(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(1,9)$, $(5,1)$, $(5,3)$, $(5,5)$, $(5,7)$, $(5,9)$, $(9,1)$, $(9,3)$, $(9,5)$, $(9,7)$, $(9,9)$, $(3,1)$, $(3,2)$, $(3,5)$, $(3,8)$, $(3,9)$, $(7,1)$, $(7,2)$, $(7,5)$, $(7,8)$, or $(7,9)$*

3. *If $n = 11$, then the deficient board is tileable if and only if the deficient square does not have coordinates $(4,11)$, $(8,11)$, $(1,8)$, $(1,4)$, $(4,1)$, $(8,1)$, $(11,4)$, or $(11,8)$*

Proof. We tested out each case with Polysolver, and these were the results that we got for the lower cases.

$\square$