SOME POSITIVE LIVSIC-LIKE RESULTS

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Abstract. In this paper, we present some results on the cohomology of cocycles over an Anosov Diffeomorphism (or shift of finite type) building on the Positive Livsic Theorem (for $\mathbb{R}$-valued cocycles). More specifically, we consider the case of an anosov diffeomorphism $f : X \to X$, with a cocycle $A : X \to \mathbb{R}$. Denoting the shift space for a Markov Partition P of X by Ω_P, and approximating the induced cocycle $A : P \to \mathbb{R}$ by locally constant cocycles on successive Markov Partitions of Ω_P, we show that some existing results can be proven using graph-theoretic techniques.

1. The graph associated to a partition of the Shift

In what follows we will work with a full shift $\sigma : \Omega_P \to \Omega_P$ given by

$$\sigma : (a_i)_{i=-\infty}^{\infty} \mapsto (a_{i+1})_{i=-\infty}^{\infty} .$$

The symbols in Omega will be assumed to be taken from a finite set of cardinality k. We define the nth partition of Ω_P to be the set of subsets (called cylinders) $C_{b_{-n}\ldots b_n} \subset \Omega_P$ defined by

$$C_{b_{-n}\ldots b_n} = \left\{ (a_i)_{i=-\infty}^{\infty} \in \Omega_P \, : \, a_i = b_i, \; -n \le i \le n \right\} .$$

Associated to the nth partition of Ω_P we can define a graph $\mathcal{G}(n)$ by associating each cylinder with a vertex, and saying there exists an edge from $C_{b_{-n}\ldots b_n}$ to $C_{b'_{-n}\ldots b'_n}$ if there exists $(a_i) \in C_{b_1\ldots b_n}$ such that $\sigma\left((a_i)\right) \in C_{b'_1\ldots b'_n}$. For convenience, we will often associate a vertex $C_{b_{-n}\ldots b_n}$ with the multi-index $B = (b_{-n}, \ldots, b_n)$.

Note that there exists a natural association between the edges of $\mathcal{G}(n)$ and the vertices of $\mathcal{G}(n+1)$. This can be clearly seen by writing an edge as an ordered pair (B, B'), and noting that $b_{i+1} = b'_i$ for all $i \le n-1$. Thus the ordered pair (B, B') is uniquely determined by the multi-index $(b_{-n}, \ldots, b_n, b'_n)$.

There are a few obvious properties of this graph that we list here, which will be useful later on. $\mathcal{G}(n)$ contains precisely k loops (that is, edges from a vertex to itself), which corresponds to the constant strings $(\ell)_{i=1}^{n}$. Similarly, there will exist $\binom{k}{2}$ pairs (A_1, A_2) of vertices connected by bidirectional edges, which will correspond to alternating sequences of pairs of symbols. We call such pairs *bidirectional pairs.* $\mathcal{G}(n)$ will consist of precisely k^{2n+1} vertices and k^{2n+1} edges. We will denote the length of our multi-indices by $m = 2n+1$

2. Markov Partitions, Shifts, and Livsic Theorems

A well known result in the study of Anosov Diffeomorphisms is the Positive Livsic Theorem, which, like several other well known theorems, relates the behavior of a cocycle around periodic points to it's cohomology. In [NP], the authors present a very general version of this theorem, which we state here without proof, an alternative statement of the theorem, with proof, can be found in [LT]. In the

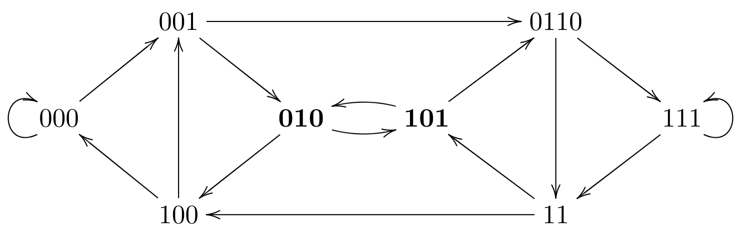

FIGURE 1.1. $\mathcal{G}(n)$ for the shift on two symbols. The bold vertices represent bidirectional pairs.

following theorem, we will use the following definitions: $\mathcal{H}_v = \{w \in \mathbb{R}^n | \langle v, w \rangle \geq 0\}$ is a half space defined by v, and $\mathcal{L}_f = \{\sum_{i=1}^{k-1} f(T^i x) | T^k x = x\}$ is the set of weights around periodic points.

Theorem 1. *Assume $T : X \to X$ is a transitive Anosov Diffeomorphism.*

(1) Let $f : X \to \mathbb{R}$ be a Holder function. If all values in $\mathcal{L}_f$ are non-negative, then f is cohomologous to a non-negative function.

(2) Let $f : X \to \mathbb{R}^n$ be a Holder function, and let $v \in \mathbb{R}^n - \{0\}$. If all values in $\mathcal{L}_f$ are included in $\mathcal{H}_v$, then f is cohomologous to a function taking values in in $\mathcal{H}_v$.

From this statement of the theorem, it is easy to see that that we can make the following generalization:

Proposition 2. *Assume $T : X \to X$ is a transitive Anosov Diffeomorphism. Let $f : X \to \mathbb{R}^n$ be a Holder function. If all values of $\mathcal{L}_f$ are included in an n-sided polygonal cone in $\mathbb{R}^n$, then f is cohomologous to a function taking values in this cone.*

Proof. Without loss of generality, we can take the cone in question to be the first orthant (this amounts to conjugating by a linear transformation). Now, projecting f onto each coordinate axis. we see that each projection $f_i : X \to \mathbb{R}$ satisfies the hypothesis of the Positive Livsic Theorem. Therefore, to each f_i we can associate a Holder $V_i : X \to \mathbb{R}$ with $f_i \geq V_i \circ T - V_i$. The function $V = (V_1, V_2, \ldots, V_n)$ will then geneare a coboundary taking f_i to a function valued in the cone. □

It is this theorem that this paper is aimed at generalizing. However, since the case of a function taking periodic data in an $n+1$ cone in $\mathbb{R}^n$ has proved mostly intractable, a new approach was required.

To simplify the problem, we take the problem to a simpler space, the shift space. This is a fairly commonly applied technique, owing to the correspondance between Markov partitions of a space equipped with an Anosov Diffeomorphism and a subshift. For a Markov Partition $\{P_i\}_{i=1}^{k}$, it is a generally known result of Bowen that we specify a point on our manifold X by a sequence in the shift space. This motivates our transition to the shift space to try and prove more general results on the cone. It should be noted that, while all of the following work is done on the full shift, it can easily be applied to subshifts.

3. Livsic-type theorems for $\mathcal{G}(n)$

We begin with a weighting $A : V_{\mathcal{G}} \to \mathbb{R}$ (where $V_{\mathcal{G}}$ is the set of vertices of $\mathcal{G}$) such that for any cycle $\eta = (v_1, \dots v_\ell)$ within $\mathcal{G}$, we have $\sum_{v\in\eta} A(v) \geq 0$. We wish to prove the following theorem:

Theorem 3. *For any weighting $A : V_{\mathcal{G}} \to \mathbb{R}$ such that for any cycle $\eta = (v_1, \dots v_\ell)$ within $\mathcal{G}$, we have $\sum_{v\in\eta} A(v) \geq 0$, we can find a weighting $T : V_{\mathcal{G}} \to \mathbb{R}$ such that*
(i) $T(x_1) - T(v) = T(x_2) - T(v)$ for any two successors x_1 and x_2 of any $v \in V_{\mathcal{G}}$
(ii) $T(x) - T(v) \leq A(v)$ for any successor x of any $v \in V_{\mathcal{G}}$

We will call any weighting such that condition (i) is satisfied a *shakespearean* weighting. Note that, by associating the weight $\hat{T}(v, x) = T(x) - T(v)$ to the edge (v, x), the weighting on vertices in Theorem 1 gives us a weighting $\hat{T}$ on edges such that, for any cycle $\eta = (v_1, v_2, \dots v_\ell)$, we have $\sum_{i=1}^{\ell} \hat{T}(v_i, v_{i+1}) = 0$.

We note here a fact, which we state as a lemma:

Lemma 4. *Any shakespearean weighting of $\mathcal{G}(n)$ over an alphabet of k symbols can be uniquely determined by k^{m-1} weights.*

Proof. Obvious. □

Since we can choose a shakespearean weighting, our proof amounts to showing that the k^m inequalities required by condition (ii) do not contradict one another. Therefore, for the proof of Theorem 1, we need one further lemma:

Lemma 5. *For any weighting $A : V_{\mathcal{G}} \to \mathbb{R}$ such that for any cycle $\eta = (v_1, \dots v_\ell)$ within $\mathcal{G}$, we have $\sum_{v\in\eta} A(v) \geq 0$, the inequalities from part (ii) of Theorem 1 can be satisfied on bidirectional pairs by a shakespearean weighting.*

Proof. On such pairs, the $k^2 - k$ associated inequalities can be rewritten as $\binom{k}{2}$ equations

$$-A(x) \leq T(x) - T(v) \leq A(v)$$

where (x, v) is a bidirectional pair. Each such inequality can only be unsolvable if

$$A(v) < -A(x)$$

but this directly contradicts the requirement that, since $\eta = (x, v)$ is a closed cycle,

$$A(v) + A(x) \geq 0$$

Therefore, each inequality is solvable individually. But, for two bidirectional pairs $\gamma_1 = (x_1, v_1)$ and $\gamma_2 = (x_2, v_2)$, there can clearly be no edges from a vertex in one to a vertex in the other. Therefore, we can determine weights on bidirectional pairs independant of the weights on any other bidirectional pair, and thus can assign a shakespearean weighting that satisfies the inequalities on these pairs. □

We now extend this solvability to the solvability of the entire system, thus proving the theorem.

Proof. (Of Theorem 1). Given inequalities which cannot simultaneously be satisfied, we can find a cycle η containing the edges corresponding to these inequalities. Note that, since all inequalities not in bidirectional pairs will involve only bounding above (ie $T_i - T_j \leq A_j$), the contradiction must involve a bidirectional pair (that is, η will contain a bidirectional pair that contradicts on of the other inequalities along the cycle).

Let η contain p vertices, and label the pair in the cycle by the numbers 1 and p. This gives us the equations

$$\begin{aligned} -A_1 &\leq T_1 - T_p \leq A_p \\ T_2 - T_1 &\leq A_1 \\ &\vdots \\ T_p - T_{p-1} &\leq A_{p-1} \end{aligned}$$

Which, simplifying until only the contradictory terms remain, yields:

$$\begin{aligned} -A_1 &\leq T_1 - T_p \leq A_p \\ T_p - T_1 &\leq \sum_{i=1}^{p-1} A_i \end{aligned}$$

The inequalities being unsolvable implies that, $-\sum_1^p A_i > A_p$, which contradicts our condition of positive sums around cycles. Therefore, since the inequalities around bidirectional pairs can be satisfied, we can assign a shakespearean weighting such that all the all the inequalities (ii) are satisfied. □

While theorems of this sort (Positive Livsic Theorems) are known for a myriad of examples, a distinct advantage of this method is it's ability to generalize to a more troublesome case. In the case of Positive Livsic Theorems, we examine cocycles whose periodic data is positive, and show them to be cohomologous to cocycles which are positive. This generalizes nicely if we replace "positive" with "lying in an ℓ sided cone in $\mathbb{R}^\ell$," however, if we wish to examine cocycles taking periodic data in an $\ell + 1$ sided cone in $\mathbb{R}^\ell$, it becomes very difficult to prove an analogue to the theorem. In the case of our graphs, we are able to prove such a result:

Theorem 6. *Let C be the $\ell + 1$ cone in $\mathbb{R}^\ell$ defined by taking all the points $(x_1, x_2 \ldots, x_\ell)$ in the first orthant such that $x_\ell \leq \sum_{i=1}^{\ell-1} \alpha_i x_i$ (where α_i are positive). Then for any weighting $A : V_{\mathcal{G}(n)} \to \mathbb{R}^\ell$ such that for any cycle $\eta = (v_1, \ldots v_\ell)$ within $\mathcal{G}(n)$, we have $\sum_{v \in \eta} A(v) \in C$, we can find a shakespearean weighting $T : V_{\mathcal{G}(n)} \to \mathbb{R}^\ell$ such that $A(v) - T(x) + T(v) \in C$ for any successor x of any $v \in V_{\mathcal{G}}$.*

Proof. As before, we need to show that some inequalities can be satisfied. By looking at A as a set of ℓ weightings $A^i : V_{\mathcal{G}(n)} \to \mathbb{R}$, we note that, by the previous theorem, we can already choose the first $\ell - 1$. When considering the ℓth coordinate, however, we have a more complex set of inequalities. If we define

$$F_{kj} = \sum_{i=1}^{\ell-1} \alpha_i (A_k^i - T_j^i + T_k^i)$$

(where subscripts denote vertices in $\mathcal{G}(n)$ and j is a successor of l), we see that our system of inequalities now becomes:

$$\begin{aligned} F_{kj} &\geq A_k^\ell - T_j^\ell + T_k^\ell \geq 0 \\ A_k^\ell - F_{kj} &\leq T_j^\ell - T_k^\ell \leq A_k^\ell \end{aligned}$$

Since our definition means that $F_{kj} \geq 0$, these inequalities are solvable so long as the special case of bidirectional pairs is solvable. Take a bidirectional pair $\eta =$

$(j,k) \in V_{\mathcal{G}(n)}$. Then we have the inequalities:

$$\begin{aligned} A_k^\ell - F_{kj} &\leq T_j^\ell - T_k^\ell \leq A_k^\ell \\ A_j^\ell - F_{jk} &\leq T_k^\ell - T_j^\ell \leq A_j^\ell \end{aligned}$$

From the proof of Theorem 1, we know that we can always find our shakespearean weighting such that

$$-A_j^\ell \leq T_j^\ell - T_k^\ell \leq A_k^\ell$$

Therefore, the solvability of this inequality reduces to the question of whether or not

$$A_k^\ell - F_{kj} \leq T_j^\ell - T_k^\ell \leq F_{jk} - A_j^\ell$$

is solvable. However by our condition on periodic data and the definition of F_{jk}, we have that

$$F_{jk} + F_{kj} = \sum_{i=1}^{\ell-1} \alpha_i (A_k^i + A_j^i) = \sum_{i=1}^{\ell-1} \alpha_i \left(\sum_{p \in \eta} A_p^i \right) \geq \sum_{p \in \eta} A_p^\ell$$

and, thus:

$$\begin{aligned} A_k^\ell + A_j^\ell &\leq F_{jk} + F_{kj} \\ A_k^\ell - F_{kj} &\leq F_{jk} - A_j^\ell \end{aligned}$$

Completing the proof. □

4. Further Directions For Research

It is important to note that, while this result is interesting and suggestive, it does not yet generalize to arbitrary cocycles on the shift in a immediate way. However, our examination of the problem suggests to us that we may be able to prove a more general result using our work on shakespearean weightings. The following proposition, in itself, suggests that, in some way, constructions on the $\mathcal{G}(n)$ can lead to more general results on the shift.

Proposition 7. *For any continuous function $f : \Omega_k \to \mathbb{R}$, the sequence of functions*

$$F_n : \mathcal{G}(n) \to \mathbb{R}$$

defined by $F_n(B) = f(b)$ for some $b \in C_B$ converges to f in the C_0 topology.

This means that we can approximate our cocycle A on the shift by locally constant functions A_n on n-cylinders.

In our examination of the graphs $\mathcal{G}(n)$, we noticed that several properties of the graphs seemed likely to lend themselves to an iterative construction of cocycles on successive graphs. Most notably, we can define a nice correspondance M between a vertex C_B in $\mathcal{G}(n)$ and vertices C_{B_i}in $\mathcal{G}(n+1)$ such that $C_{B_i} \subset C_B$, as shown in figure 4.1. Under this correspondance, it is easy to see that the graph $\mathcal{G}(n)$ is a topological minor of the graph $\mathcal{G}(n+1)$.

Using this correspondance, we suspect that we could build a cauchy sequence T_n of shakespearean coboundaries to the $A_n$defined using Proposition 5. This would allow us to construct, as the limit of this sequence, a continuous coboundary to our cocycle with the desired properties.

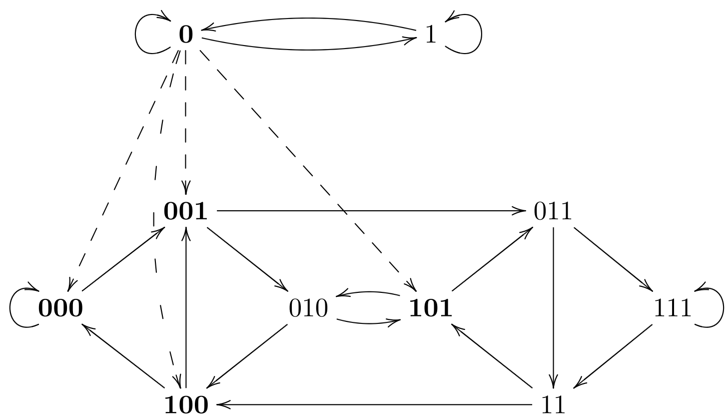

FIGURE 4.1. A map of the correspondence M between the vertex 0 in $\mathcal{G}(0)$ and vertices in $\mathcal{G}(1)$. While this correspondance holds for all n, $\mathcal{G}(n)$ is not planar for $n > 1$, meaning that the graph becomes harder to visualize.

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