THE STEINHAUS CONDITION FOR SETS IN $\mathbb{R}^n$ AND $\mathbb{U}$

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Abstract. We prove that every set in the Urysohn universal space, $\mathbb{U}$, has an isometric copy which is Jackson within $\mathbb{U}$. We also put forward the conjectures that every finite (or compact) subset of $\mathbb{R}^n$ has an isometric embedding within some $\mathbb{R}^N$ which is Jackson.

Contents

In this note, we describe a conjecture made of finite and compact metric spaces regarding their relation to a condition of Steinhaus. In section 1, we give brief definitions and examples, and state our Conjectures. In section 2, we define the Urysohn universal space $\mathbb{U}$, give a few of its properties, and introduce the notion of a Katětov map on a metric space. In section 3, we prove that every set within $\mathbb{U}$ is essentially Jackson; that is, every set within $\mathbb{U}$ is isometric to a Jackson set in $\mathbb{U}$. In section 4, we describe the connection between $\mathbb{U}$ and the Conjectures made in section 1.

1. Definition and Basic Examples

The condition in question was first described in a question made in the 1950s regarding the lattice $\mathbb{Z}^2$ within $\mathbb{R}^2$.

Question 1.1 (Steinhaus). *Does there exist a set $S \subset \mathbb{R}^2$ such that for every isometry $\phi \in \mathrm{Iso}(\mathbb{R}^2)$, we have that $|\phi(S) \cup \mathbb{Z}^2| = 1$?*

This question was answered in the affirmative by Jackson and Mauldin in [3]. More generally, we may define the following notions:

Definition 1.2. Let U be a metric space. A set $X \subseteq U$ *has a Steinhaus set within U* if there exists a set $S \subseteq U$ such that for all isometries $\phi \in \mathrm{Iso}(U)$, we have that $|\phi(X) \cap S| = 1$. The set S is a *Steinhaus set for X within U.*

Definition 1.3. Let U be a metric space, A set $X \subset U$ is *Jackon within U* if X has no Steinhaus set within U.

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The main conjectures of this note, then, are the following:

Conjecture 1.4. *Let $X \subset \mathbb{R}^n$ be a finite metric space. Then, there exists $N \in \mathbb{N}$ such that $X \hookrightarrow \mathbb{R}^N$ and X is Jackson inside $\mathbb{R}^N$.*

More generally,

Conjecture 1.5. *Let $X \subset \mathbb{R}^n$ be compact. Then, there exists $N \in \mathbb{N}$ such that $X \hookrightarrow \mathbb{R}^N$ and X is Jackson inside $\mathbb{R}^N$.*

To give a brief intuition of the above notion, note that it is dependent both on the set X and the ambient space U, but is invariant under any map $f : X \to X$ that preserves the metric up to a non-negative constant. For conciseness, we will suppress the notation *"within U"* when the ambient space is clear from context.

Example 1.6. There are easy examples of sets which, for instance, have a Steinhaus set within $\mathbb{R}$ but are Jackson within $\mathbb{R}^2$. Any two-point set, for example, is similar to the set $X = \{0, 1\} \subset \mathbb{R}$, and the set $S = \bigcup_{k\in\mathbb{Z}}[2k, 2k + 1)$ is Steinhaus for X. Yet for any isometric embedding $\tilde{\cdot} : X \hookrightarrow \mathbb{R}^2$, we can see that $\tilde{X}$ is Jackson within $\mathbb{R}^2$ as follows: suppose that S were a Steinhaus set for $\tilde{X}$ and, without loss of generality, that $\tilde{0} \in S$. Then, $\tilde{1} \notin S$; moreover, for any isometry $\phi \in \mathrm{Iso}(\mathbb{R}^2)$, $\phi(\tilde{1}) \notin S$. However, the space $\mathbb{R}^2$ is *metrically homogeneous* in the sense of the following:

Definition 1.7. Let U be a metric space. Then, U is *metrically homogeneous* if for any two finite subsets $X, Y \subseteq U$, every isometry $\psi : X \to Y$ extends to an isometry $\hat{\psi} : U \to U$.

A very detailed analysis of the properties of such spaces is given in [2].

Then, we may find a contradiction in at least two ways:

First, we may infer that $\partial B_1(\tilde{0}) \bigcap S = \emptyset$. Yet there exist points $u, v \in \partial B_1(\tilde{0})$ such that $d(u, v) = 1$, so that there exists an isometry $\psi : \{\tilde{0}, \tilde{1}\} \to \{u, v\}$; since $\mathbb{R}^2$ is metrically homogeneous, ψ extends to an isometry $\hat{\psi}$ on U. But then $\hat{\psi}(\tilde{X}) \cap S = \emptyset$, a contradiction to Definition 1.2.

Alternatively, let $\tilde{0}'$ be the reflection of $\tilde{0}$ over $\tilde{1}$. Then, because $\tilde{1} \notin S$ and there is an isometry $\{\tilde{0}, \tilde{1}\} \xrightarrow{\psi} \{\tilde{0}', \tilde{1}\}$ that fixes $\tilde{1}$ which extends to an isometry on $\mathbb{R}^2$, it must be that $\tilde{0}' \in S$. Yet, now we may infer $\partial B_2(\tilde{0}) \subset S$; however, there are now points $u, v \in \partial B_2$ isometric to $\tilde{X}$, contradicting the condition on S.

For finite (and compact) sets $X \subset \mathbb{R}^n$, the two arguments above, which we will call the rotation argument and the reflection argument, respectively, appear to be useful in the proof that X is Jackson. The arguments rely on the additional "room" within the ambient space; that is, they rely on the isometries which fix *all but one point of X*, and on the shape of the orbit of that distinguished point. For instance, a set $X \subset \mathbb{R}^n$ is easily seen to be Jackson if all but one of its points are contained in an affine subspace, by the reflection argument.

2. Considerations in the Urysohn Universal Space

We have found it useful to consider isometric copies of the finite spaces and of $\mathbb{R}^n$ sitting within the Urysohn universal space:

Definition 2.1. The *Urysohn universal space*, $\mathbb{U}$, is the unique (up to isometry) complete and separable (i.e. Polish) metric space with the following two properties:

(1) $\mathbb{U}$ is metrically homogeneous, and
(2) every separable metric space X embeds isometrically into $\mathbb{U}$.

As noted in [4] and proven in [1], $\mathbb{U}$ actually satisfies a stronger property than metric homogeneity:

Proposition 2.2. *For any two compact subsets $X, Y \subseteq \mathbb{U}$, every isometry $\psi : X \to Y$ extends to an isometry $\hat{\psi} : U \to U$.*

We will call this property *compact-homogeneity*. We note that $\mathbb{R}^n$ is compact-homogeneous as well.

For brevity's sake, we will not give the construction of $\mathbb{U}$, which can be found in [4], [5], [6], and in many other sources on $\mathbb{U}$. We only note that the basic tool with which $\mathbb{U}$ is constructed is the following class of functions:

Definition 2.3. Let X be a metric space. Then, a function $f : X \to \mathbb{R}$ is a *Katětov function* if for all $x, y \in X$,

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

The class of Katětov functions is simply the class of distance functions to points in metric extensions of X. In other words, a function f is a Katětov function if and only if there exists a metric space Y containing X, and a point $y \in Y$, such that $d(x, y) = f(x)$ for all $x \in X$. From this, it is clear that if f is a Katětov function on X and $f(x) \neq 0$ for all $x \in X$, then f yields a one-point metric extension of X.

We will look at $\mathbb{U}$ in two directions: both as a tool for our work towards Conjectures 1.4 and 1.5, and as an ambient space itself.

3. Every Subset of $\mathbb{U}$ Is Essentially Jackson

In this section, we prove the following result:

Theorem 3.1. *Let $X \subseteq \mathbb{U}$, with $|X| > 1$. Then, X is* essentially Jackson*; that is, there exists an isometric copy of X which is Jackson within $\mathbb{U}$.*

We will rely on a result from [6]:

Lemma 3.2. *Let F be a Polish space. Then, there exists an isometric embedding $I : F \hookrightarrow \mathbb{U}$ and a monomorphism of groups $I^* : \mathrm{Iso}(F) \hookrightarrow \mathrm{Iso}(\mathbb{U})$ such that*

$$I(\phi(x)) = I^*(\phi)(x) \quad \forall x \in F,\ \forall \phi \in \mathrm{Iso}(F),$$

that is, for all $\phi \in \mathrm{Iso}(F)$, the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, hook, "I"] \arrow[d, "\phi"] & \mathbb{U} \arrow[d, "I^*(\phi)"] \\ F \arrow[r, hook, "I"] & \mathbb{U} \end{tikzcd}$$

We now prove Theorem 3.1:

Proof. Let $D(X) = \{d(x, y) | x, y \in X\}$ be the distance set of X. Fix $x_0 \in X$ and a non-zero $c \in D(X)$. Define $f : X \to \mathbb{R}$ by $f(x) = \max\{d(x_0, x), c\}$ for all $x \in X$. It is easy to check that f is a nowhere-zero Katětov function on X taking values only in $D(X)$, so that it defines a one-point metric extension of X with the same distance set.

Now, let $X' = X \cup \{x', x''\}$ be the metric space with the following metric:

(1) $d_{X'}(x,y) = d_X(x,y) \quad \forall x,y \in X$;
(2) $d_{X'}(x,x') = d_{X'}(x,x'') = f(x) \quad \forall x \in X$;
(3) $d_{X'}(x',x'') = c$;
(4) $d_{X'}(x',x') = d_{X'}(x'',x'') = 0$.

It is easy to check that $d_{X'}$ indeed defines a metric on X', and it is clear that $D(X) = D(X \cup \{x',x''\}) = D(X')$. Since $X \subseteq \mathbb{U}$ is separable, X' is separable, so that $\bar{X}' = \bar{X} \cup \{x',x''\}$ is Polish. By Lemma 3.2, there is an isometric embedding $I : \bar{X}' \hookrightarrow \mathbb{U}$ and a monomorphism of groups $I^* : \mathrm{Iso}(\bar{X}') \hookrightarrow \mathrm{Iso}(\mathbb{U})$ as above. (It is interesting to note that possibly $\bar{X}' \supsetneq \mathbb{U}$, yet $\bar{X}'$ still embeds in this way into $\mathbb{U}$.)

Note that $X \cup \{x'\}$ and $X \cup \{x''\}$ are isometric, so that $\bar{X} \cup \{x'\}$ and $\bar{X} \cup \{x''\}$ are isometric as well. Let $\phi \in \mathrm{Iso}(\bar{X}')$ be the isometry that fixes $\bar{X}$ (in particular, fixes X) and sends $x' \mapsto x''$. For clarity, let $\hat{\phi} = I^*(\phi)$.

We now show that $I\left(X \cup \{x'\}\right)$ is Jackson. Suppose S is a Steinhaus set for $I\left(X \cup \{x'\}\right)$, and without loss of generality, we may assume that $I(x') \in S$. Then, because $\hat{\phi}$ fixes $I(X)$, we must have that $I(x'') = I(\phi(x')) = \hat{\phi}(I(x')) \in S$. But $d(I(x'), I(x'')) = c \in D(X)$, so there exist $x,y \in I(X)$ such that $d(x,y) = c$. Thus, since $\mathbb{U}$ is metrically homogeneous, there exists an isometry $\psi \in \mathrm{Iso}(\mathbb{U})$ such that $x \overset{\psi}{\mapsto} x'$ and $y \overset{\psi}{\mapsto} x''$. But this implies that S meets $\psi\left(I\left(X \cup \{x'\}\right)\right)$ at two points, a contradiction.

Yet we wish to show $I(X)$ is Jackson; we will be done once we have the following lemma:

Lemma 3.3. *Let U be a metrically homogeneous space. Suppose $Y \subset Y' \subset U$, with $|Y| > 1$, and suppose $D(Y') = D(Y)$. If Y' is Jackson, then Y is Jackson.*

The proof of Lemma 3.3 is quick: Let $S \subseteq U$. If $Y' \cap S = \emptyset$, then certainly $Y \cap S = \emptyset$. If $\{y_1', y_2'\} \subseteq Y' \cap S$, then by the hypothesis, there exist $y_1, y_2 \in Y$ such that $d(y_1,y_2) = d(y_1',y_2')$. Then, since U is metrically homogeneous, the function sending $y_1 \overset{\varphi}{\mapsto} y_1'$ and $y_2 \overset{\varphi}{\mapsto} y_2'$ extends to $\hat{\varphi} \in \mathrm{Iso}(U)$. But this implies that $\{y_1,y_2\} \subseteq \hat{\varphi}(Y) \cap S$, which finishes the proof.

Applying Lemma 3.3 to $I\left(X \cup \{x'\}\right)$, we conclude that $I(X)$ is Jackson within $\mathbb{U}$, as desired. □

In particular, because $\mathbb{U}$ is compact-homogeneous,

Corollary 3.4. *Let $X \subsetneq \mathbb{U}$ be compact, and suppose $|X| > 1$. Then, X is Jackson.*

It is still unclear whether, for non-compact proper subsets of $\mathbb{U}$, essentially Jackson $\implies$ Jackson; this would follow if $\mathbb{U}$ were completely homogeneous (i.e. if for *any* subsets $A, B \subset U$, an isometry $A \to B$ extends to an isometry on $\mathbb{U}$). This seems unlikely, however, in light of the fact that no isometry $\mathbb{U}_0 \to \mathbb{U}$, where $\mathbb{U}_0 \subsetneq \mathbb{U}$ is an isometric copy of $\mathbb{U}$, can extend to an isometry on the whole space.

However, Theorem 3.1 establishes that, except in the trivial cases, the possession of a Steinhaus set is not at all an intrinsic property of a Polish space: the ambient space $\mathbb{U}$ will always have enough "room" for an isometric copy of the Polish space to be Jackson. This fact strongly suggests Conjectures 1.4 and 1.5.

4. Discussion of Steinhaus Condition for Sets in $\mathbb{R}^n$

Returning to sets in $\mathbb{R}^n$, we have the following proposition:

Proposition 4.1. *Let $X \subset \mathbb{R}^n$, and let $i : \mathbb{R}^n \hookrightarrow \mathbb{U}$ be an isometric embedding. If X is Jackson, then $i(X)$ is Jackson.*

Proof. Suppose X is Jackson, but $S \subseteq \mathbb{U}$ is Steinhaus for $i(X)$. Consider $S \cap i(\mathbb{R}^n) \subset \mathbb{U}$. We know that $i^{-1}(S \cap i(\mathbb{R}^n))$ is not Steinhaus for X, so there exists an isometry $\phi \in \mathrm{Iso}(\mathbb{R}^n)$ such that

$$|\phi(x) \cap i^{-1}(S \cap i(\mathbb{R}^n))| \neq 1$$

By Lemma 3.2, ϕ extends to $\hat{\phi} \in \mathrm{Iso}(\mathbb{U})$, and the diagram

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook, "i"] \arrow[d, "\phi"] & \mathbb{U} \arrow[d, "\hat{\phi}"] \\
\mathbb{R}^n \arrow[r, hook, "i"] & \mathbb{U}
\end{tikzcd}$$

commutes, i.e. $i(\hat{\phi}(x)) = \phi(i(x))$ for all $x \in \mathbb{R}^n$. Now,

$$\begin{aligned}
i\left(\phi(X) \cap i^{-1}\left(S \cap i(R^n)\right)\right) &= \hat{\phi}\left(i(X)\right) \cap \left(S \cap i(\mathbb{R}^n)\right) \\
&= \hat{\phi}(i(X)) \cap S,
\end{aligned}$$

since $\hat{\phi}(i(X)) \subset i(\mathbb{R}^n)$. But this implies that $|\hat{\phi}(i(X)) \cap S| \neq 1$, contradicting the assumption that S is Steinhaus for $i(X)$. □

Now, because of the strength of Theorem 3.1, a weakened version of the reverse direction of Proposition 4.1 would imply Conjectures 1.4 and 1.5. That is, we conjecture that:

Conjecture 4.2. *Let $X \subset \mathbb{R}^n$ be finite, and let $i : X \hookrightarrow \mathbb{U}$ be an isometric embedding. If $i(X)$ is Jackson, then there exists $N \in \mathbb{N}$ such that an isometric copy of X is Jackson within $\mathbb{R}^N$.*

Conjectures 1.4 and 1.5 would follow.

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