Research projects

June 30, 2008

What follows is a list of projects that will be offered to the REU participants. This list is not complete: other projects will be added later, in particular, related to the mini-course offered to the participants. The order of project is random.

1 Commuting billiard ball maps

Given a convex plane domain, the billiard ball map sends the incoming ray (the trajectory of the billiard ball) to the outgoing one: the law of reflection is "the angle of incidence equals the angle of reflection".

Consider two nested convex domains. Then one has two billiard ball maps, T_1 and T_2, acting on the oriented lines that intersect both domains. Assume that the two maps commute: $T_1 \circ T_2 = T_2 \circ T_1$. **Conjecture**: *the two domains are bounded by confocal ellipses.*

If the domains are bounded by confocal ellipses then the respective billiard ball maps commute; see [33] or [35]. For outer (a.k.a. dual) billiards a similar (but perhaps easier) fact is proved in [32].

2 Every plane billiard has a 2- or a 3-periodic billiard trajectory

More precisely, the problem is to show that *every bounded plane domain with smooth boundary has a 2- or 3-periodic billiard trajectory.* This fact is established in [5] using very non-elementary methods. Is there a direct geometric proof?

There are easy examples of (necessarily, non-convex) domains without 2-periodic billiard trajectories or without 3-periodic ones – see [35].

3 Can one-parameter families of 2- and 3-periodic billiard trajectories coexist?

A curve of constant width admits a one-parameter family of 2-periodic (back and forth) billiard trajectories. Likewise, for every $p \geq 3$, there exist billiard tables admitting a one-parameter family of p-periodic billiard trajectories; see [1, 19, 40] for a recent approach using ideas of sub-Riemannian geometry.

Problem: *are there smooth convex curves, other than ellipses, simultaneously admitting one-parameter families of p- and q-periodic billiard trajectories (for $p \neq q$)?* The simplest case of the question is whether any curve of constant width, other than a circle, admits a one-parameter family of 3-periodic billiard trajectories.

A similar question can be asked about outer billiards.

4 Periodic orbits for polygonal outer billiards in the hyperbolic plane

The outer billiard about a convex polygon P is a piece-wise isometry of the exterior of P defined as follows: given a point x outside of P, find the support line to P through x having P on the left, and reflect x in the support vertex.

C. Culter proved at Penn State REU 2004 that every polygon admits periodic outer billiard orbits [37]. Outer billiard can be defined on the sphere and in the hyperbolic plane. On the sphere, there exist polygons without periodic outer billiard orbits. **Conjecture**: *every polygonal outer billiard in the hyperbolic plane has periodic orbits.* These orbits may lie on the circle at infinity.

See [13, 14] for outer billiards in the hyperbolic plane. For recent advances in the very hard problem whether every polygonal billiard admits periodic trajectories, see R. Schwartz's web site www.math.brown.edu/∼res/.

5 Complexity of polygonal outer billiards

The complexity of a language is a function $p(n)$ equal to the number of different n-letter words in the language. Given a convex k-gon P in the Euclidean plane, label its vertices by $a_1, \ldots, a_k$. The orbit of every point is then encoded by an infinite word $\ldots, a_{i_1}, a_{i_2}, \ldots$ consisting of the labels of the vertices at which the consecutive reflections occur. This results in a language with the alphabet $a_1, \ldots, a_k$, and the resulting complexity $p(n)$ is called the complexity of the outer billiard about P.

It is known [22] that the complexity of polygonal outer billiards has polynomial growth (that is, $p(n)$ is bounded above by a polynomial in n), and that if P is a lattice polygon then $p(n)$ has quadratic lower and upper bounds. According to a recent theorem by N. Bedaride, the complexity is also quadratic for the regular pentagon. *What can be said about other polygons?* For example, one may consider the class of quadrilaterals called kites. R. Schwartz has a powerful program BilliardKing dealing with the kites; it is downloadable from his web site.

6 An analog of Birkhoff's theorem for Lorentz billiards

The classical Birkhoff theorem states that, for every $n \geq 3$ and $1 \leq k \leq n/2$, the billiard system inside a plane oval has at least two n-periodic trajectories with the rotation number k, see, e.g., [35]. Consider the billiard system inside an oval in the Lorentz plane with the pseudo-Euclidean metric $ds^2 = dx^2 - dy^2$. *Is there an analog of Birkhoff's theorem in this set-up?*

See [24] for Lorentz billiards. Note that, unlike the Riemannian case, there may be no geodesics connecting two points of a pseudo-Riemannian manifold.

7 A dynamical system on an oval

Let γ be a plane oval and (u, v) a pair of directions. Define a mapping $T_{(u,v)} : \gamma \to \gamma$ as follows: given a point $x \in \gamma$, draw the line in the direction u through x, find its second intersection point y with γ, draw the line in the direction v through y, and let z be its second intersection point with γ; set

$T_{(u,v)}(x) = z$. The map $T_{(u,v)}$ was studied in different contexts: in relation to Hilbert's 13th problem, the Sobolev equation for fluid oscillations in a fast rotating tank, the theory of Lorentz surfaces.

If γ is an ellipse then $T_{(u,v)}$ is conjugated to a circle rotation for every choice of the directions (u, v). **Conjecture**: *let γ be a plane oval such that, for every pair of directions u and v, the map $T_{(u,v)}$ is conjugated to a circle rotation. Then γ is an ellipse.* See [18] for detailed discussion, references and a weaker result.

8 Closure condition for a chain of null geodesics on an ellipsoid

The following Poncelet-style theorem was proved in [18]. Consider an ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \quad a, b, c > 0$$

in three dimensional Minkowski space with the metric $dx^2 + dy^2 - dz^2$. The induced metric on the ellipsoid degenerates along the two "tropics"

$$z = \pm c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}};$$

the metric is Lorentz (of signature $(+, -)$) in the "equatorial belt" bounded by the tropics. Through every point of the equatorial belt there pass two null geodesics of the Lorentz metric, the "right" and the "left" ones.

Call a chain of alternating left and right null geodesics, going from tropic to tropic, an (n, r)-chain if it closes up after n steps and making r turns around the equator. The theorem states that if there exists an (n, r)-chain of null geodesics then every chain of null geodesics is an (n, r)-chain. See [18] for a discussion.

Problem: *find conditions on the numbers a, b, c ensuring the existence of (n, r)-chains.* For the classical Poncelet porism, such a formula is due to Cayley, see [21]. For the Poncelet porism, see [7] and [2].

9 Closure condition in the zig-zag theorem and its generalizations

The zig-zag theorem [6, 2] concerns two circles in Euclidean 3-space positioned in such a way that, for some number d, each point of either circle is distance d from exactly two points of the other circle (this is not too restrictive). Consider a chain of points $x_1, x_2, \dots$ so that even points lie on one circle, odd points on another circle and $|x_i - x_{i+1}| = d$ for all i. The claim is that if one such chain closes up after n steps then so does every such chain.

Problem: *find conditions on the circles and d ensuring that the chain closes after n steps making k turns around the circles.*

In a very particular case, when the two circles are replaced by intersecting lines, the situation is equivalent to the system described in 7 with the oval being an ellipse. The zig-zag configuration closes up if and only if the angle between the lines is π-rational.

Another problem is to generalize the zig-zag theorem to two circles in Euclidean spaces of dimension 4 and 5, and to the spherical or hyperbolic spaces.

10 A converse Desargues theorem

A classical Desargues theorem states the following. Consider a pencil of conics (a one-parameter family of conics sharing four points – these points may be complex or multiple, as for the family of concentric circles). The intersections of a line ℓ with these conics define an involution on ℓ, and the theorem states that this involution is a projective transformation of ℓ. See [4].

Let $f(x, y)$ be a (non-homogeneous) polynomial with a non-singular value 0. Let γ be an oval which is a component of the algebraic curve $f(x, y) = 0$. Assume that the curves $\gamma_\varepsilon = \{f(x, y) = \varepsilon,\ \varepsilon > 0\}$ foliate an outer neighborhood of γ and that for every tangent line ℓ to γ, its intersections with the curves γ_ε define a (local) projective involution on ℓ. **Problem**: *prove that γ is an ellipse and the curves γ_ε form a pencil of conics.*

A particular case, in which the involutions under consideration are central symmetries of the line, is proved in [39].

11 Polyhedral outer billiards in 4-dimensional space

Let M be a closed convex hypersurface in $\mathbf{C}^2$. For a point $x \in M$, let $n(x)$ be the outer unit normal vector. The *outer billiard map* T of the exterior of M is defined as follows. For $t > 0$, consider points $y = x + \sqrt{-1}tn(x)$ and $z = x - \sqrt{-1}tn(x)$; then $T(y) = z$. One can prove that for every y outside of M there exists a unique $x \in M$ and $t > 0$ such that $y = x + \sqrt{-1}tn(x)$, hence the map T is well defined. See [33, 35, 14] for more details.

Problem: *study the dynamics of the outer billiard map when M is the surface of a convex polyhedron in $\mathbf{C}^2$, especially, when M is a regular polyhedron.*

In the plane, a regular pentagon (and other regular n-gons with $n \neq 3, 4, 6$) yields a beautiful fractal set, the closure of an infinite orbit of the outer billiard map; see the references above.

12 Geometry of bicycle curves and bicycle polygons

A closed curve γ in a Riemannian manifold M is called *bicycle* if two points x and y can traverse γ in such a way that the arc length xy remains constant and so does the distance between x and y in M. For example, a circle in Euclidean plane is a bicycle curve.

An explanation of the terminology is as follows. Let γ be a bicycle curve in the plane and let γ' be the envelope of the segments xy. If γ and γ' are the front and rear bicycle wheel tracks, then one cannot determine the direction in which the bicycle went from these curves. See [36] for a detailed discussion and references. Surprisingly, a bicycle curve can be also characterized as the section of a homogeneous cylinder (a log) that float in equilibrium in all positions.

Call the ratio of the arc length xy to the total arc length of γ the *rotation number* and denote it by ρ. In the plane, one can prove that, for some values of ρ (for example, $\rho = 1/3$ and $1/4$), the only bicycle curve is a circle, whereas for some other values (for example, $\rho = 1/2$), there exist non-circular bicycle curves; see [8, 36, 41, 42, 43]. A complete description of plane bicycle curves is not known.

Problem: *construct non-trivial examples of bicycle curves on the sphere, the hyperbolic plane, multi-dimensional Euclidean space, etc.*

A discrete analog of a bicycle curve is a bicycle (n,k)-gon, an equilateral n-gon whose k-diagonals are all equal to each other. For some values of (n,k) such a polygon is necessarily regular and for other values (say, $n=8, k=3$) non-trivial examples exist. A complete description is not known either; see [9, 11, 36] for partial results.

13 A bicycle making a unicycle track

A mathematical model of a bicycle is an oriented unit segment AB in the plane that can move in such a way that the trajectory of the rear end A is always tangent to the segment. Sometimes the trajectories of points A and B coincide (say, riding along a straight line).

The following construction is due to D. Finn [15]. Let $\gamma(t),\ t \in [0,L]$ be an arc length parameterized smooth curve in the plane which coincides with all derivatives, for $t=0$ and $t=L$, with the x-axis at points $(0,0)$ and $(1,0)$, respectively. One uses γ as a "seed" trajectory of the rear wheel of a bicycle. Then the new curve $\Gamma = T(\gamma) = \gamma + \gamma'$ is also tangent to the horizontal axis with all derivatives at its end points $(1,0)$ and $(2,0)$. One can iterate this procedure yielding a smooth infinite forward bicycle trajectory $\mathcal{T}$ such that the tracks of the rear and the front wheels coincide.

It is proved in [27] that the number of intersections of each next arc of $\mathcal{T}$ with the x-axis is greater than that of the previous one. Likewise, the number of local maxima and minima of the height function y increases with each step of the construction.

Conjecture: *Unless γ is a straight segment, the amplitude of the curve $\mathcal{T}$ is unbounded, i.e., $\mathcal{T}$ is not contained in any horizontal strip, and $\mathcal{T}$ is not embedded, that is, it starts to intersect itself.*

14 Bicycle track monodromy

The oriented trajectory of the rear bicycle wheel uniquely determines the trajectory of the front wheel, whereas the latter determines the former only after the initial position of the bicycle is chosen. For a given trajectory of the front wheel γ, one obtains the *monodromy map* $T_\gamma : S^1 \to S^1$ that assigns the terminal position of the bicycle to its initial position.

It is proved in [16] that, for every γ, the monodromy T_γ belongs to the group of isometries of the hyperbolic plane acting on S^1 as the circle at infinity; see also [27] for a multi-dimensional version. Isometries of H^2 can be elliptic, parabolic and hyperbolic [3]; their action on S^1 has no, one or two fixed points, respectively.

In [27], we proved the following Menzin's conjecture: if γ is a closed convex curve bounding area greater than π then the respective monodromy is hyperbolic.

Problem: *extend these results to bicycle motion on the sphere and the hyperbolic plane.*

15 Self-dual curves and surfaces

Projective duality is a correspondence between points of the real projective plane $\mathbf{RP}^2$ and lines of the dual projective plane $(\mathbf{RP}^2)^*$; projective duality extends to smooth and piece-wise smooth curves, taking a curve $\gamma \subset \mathbf{RP}^2$ (a one-parameter family of points) to the envelope $\gamma^* \subset (\mathbf{RP}^2)^*$ of the respective one parameter family of dual lines. A curve γ is called projectively self-dual if there exists a projective transformation from $\mathbf{RP}^2$ to $(\mathbf{RP}^2)^*$ that takes γ to γ^*. Likewise one defines self-dual polygons.

The general problem of describing projectively self-dual curves is poorly understood, see [17] for a description of projectively self-dual polygons and some results on self-dual curves.

Problem: *extend the results of [17] to projectively self-dual hypersurfaces and polyhedra, and to projectively self-dual non-degenerate curves and polygons, in multi-dimensional projective spaces.*

A non-degenerate curve γ in $\mathbf{RP}^n$ has the osculating hyperplane at each point; a hyperplane in $\mathbf{RP}^n$ is a point in the dual projective space $(\mathbf{RP}^n)^*$, and this one-parameter family of points in $(\mathbf{RP}^n)^*$ is the dual curve γ^*.

All these problems have affine analogs in which the curves (hypersurfaces) are assumed to be star-shaped and the projective duality is replaced by the polar duality.

16 A characterization of projectively regular polygons

The following problem was communicated by Richard Schwartz. Consider a convex octagon P and draw all its 3-diagonals (connecting ith vertex to $i+3$rd one). If P is sufficiently close to a regular one, these diagonals form a new convex octagon, P_1. Apply the construction to P_1, if possible, etc.

Problem: *prove that if the this operation can be applied infinitely many times then P is projectively regular.*

Of course, a similar question can be asked about other polygons and different choices of diagonals.

17 Converse 4- and 6-vertex theorems

The classical four vertex theorem of Mukhopadhyaya states that the curvature of a plane oval has at least four extrema (see, e.g., [30]). A converse theorem, due to H. Gluck, states that a positive periodic function with at least four extrema is the curvature of a convex closed parameterized plane curve, see [12]. See [38] for other similar converse theorems.

Problem 1. Another classic theorem is that a plane oval has at least six affine vertices. An affine vertex is a point at which the curve is abnormally well approximated by a conic: at a generic point, a conic passes through five infinitesimally close points of the curve, whereas at an affine point, this number equals six. Every oval γ can be given an *affine parameterization* such that $\det(\gamma'(x), \gamma''(x))$ is constant. Then $\gamma'''(x) = -k(x)\gamma'(x)$ where the function $k(x)$ is called the affine curvature. The affine vertices are the critical points of the affine curvature, see [30].

A conjectural converse theorem asserts that *if a periodic function $k(x)$ has at least six extrema then there exists a plane oval $\gamma(x)$ whose affine curvature at point $\gamma(x)$ is $k(x)$.*

Problem 2. The four vertex theorem has numerous discrete versions, see [30] for surveys and references. For example, let P be a convex n-gon with vertices $x_1, \ldots, x_n$. Assume that $n \geq 4$ and that no four consecutive vertices lie on a circle. Consider the circles circumscribing triples of consecutive vertices $x_{i-1}x_ix_{i+1}$, and assume that the center of this circle lies inside the cone of the vertex x_i (such a polygon is called *coherent*). Let $r_1, \ldots, r_n$ be the cyclic sequence of the radii of the circles. Then the sequence $r_1, \dots, r_n$ has at least two local maxima and two local minima.

A conjectural converse theorem asserts that *if a cyclic sequence* $r_1, \dots, r_n$ *has at least two local maxima and two local minima then it corresponds, as described above, to a coherent convex polygon.*

Another version of discrete four vertex theorem concerns the circles tangent to the triples of consecutive sides of a polygon: the radii of such inscribed circles also form a cyclic sequence with at least two local maxima and two local minima. One conjectures that a converse theorem holds as well.

18 Areas and partitions into triangles

If a square is partitioned into triangles of equal areas then the number of these triangles must be even. This theorem has a surprisingly difficult proof, see [31]. There are many similar results (proved along the same lines), for example, the square can be replaced by any centrally symmetric polygon [28].

An attempt to understand this result leads to the following question. Consider a partition of a square into n triangles. We allow small perturbation of triangles, so that each "free" vertex has two degrees of freedom and each vertex on a side has one degree of freedom. Denote by M the "moduli space" of partitions with the same combinatorics, i.e., obtained from a given one by small perturbations, so that no triangle degenerates. *Claim*: the dimension of M equals $n-2$ (Proof: compute the sum of angles of all the triangles...)

Assign to every partition in M the ordered set of areas of its triangles. This gives a map $D^{n-2} \to \mathbf{R}^n$; this map is component-wise quadratic (the area of triangle is given by a determinant). The image lies in the hyperplane $a_1 + \cdots + a_n = 1$ (assuming that the square is unit). Therefore there exists another polynomial relation between the areas $a_1, \dots, a_n$ depending only on the combinatorics of the partition.

Problem: *what can be said about this other polynomial relation? For example, how to find the least degree of the polynomial in terms of the combinatorics of the partition?*

A previous Penn State REU project [23] provided a description of the partitions for which this relation is linear.

The problem has to do with elimination theory: each area is a quadratic polynomial in the coordinates of all the vertices involved, and the goal is to eliminate these coordinates. See somewhat similar work on the "bellows conjecture" expressing the volume of a polyhedron in terms of its combinatorics and edge lengths [10].

Finally, note that the problem has multi-dimensional versions (motivated by the fact that if an n-dimensional cube is partitioned into simplices of equal volumes then the number of simplices is divisible by $n!$). Another variation replaces the square by a flat torus; two partitions are considered as the same if one is a parallel translation of the other.

19 DNA geometric inequality in 3-space

The average absolute curvature of a closed curve is the integral of the absolute value of its curvature with respect to the arc length parameter divided by the total perimeter length. The DNA geometric inequality states that if Γ is a closed convex plane curve and γ is a closed (maybe self-intersecting) curve inside Γ then the average absolute curvature of γ is not less than that of Γ (the configuration of two curves resembles DNA inside a cell). See [25, 26, 29, 34].

Problem: *extend the DNA inequality to 3-dimensional space.* There are at least two cases: γ is either a closed curve or a closed surface inside a convex surface Γ. If Γ is a round sphere, the result is known (in both cases).

20 Is there a totally skew 3-dimensional disc in 7-dimensional space?

A submanifold $M^k \subset \mathbf{R}^n$ is called *totally skew* if, for every two distinct points $x, y \in M$, the tangent spaces at these points are in general position (i.e., their affine span has dimension $2k+1$). Clearly, a necessary condition for being totally skew is $n \geq 2k+1$. It is proved in [20] that if M^k is a totally skew disc in $\mathbf{R}^{2k+1}$ then $k \in \{1, 3, 7\}$. For $k = 1$, a simple example is given by the cubic curve $(t, t^2, t^3),\ t \in \mathbf{R}$.

Problem: *is there a totally skew 3-disc in* $\mathbf{R}^7$*?*

The numbers $1, 3, 7$ appears in mathematics very often; they suggest that the skew field of quaternions has to do with this problem.

References

[1] Yu. Baryshnikov, V. Zharnitsky. *Sub-Riemannian geometry and periodic orbits in classical billiards.* Math. Res. Lett. 13 (2006), no. 4, 587–598.

[2] W. Barth, Th. Bauer. *Poncelet theorems.* Exposition. Math. 14 (1996), 125–144.

[3] A. Beardon. *The geometry of discrete groups.* Springer-Verlag, 1983.

[4] M. Berger. *Geometry.* Springer-Verlag, 1987.

[5] V. Benci, F. Giannoni. *Periodic bounce trajectories with a low number of bounce points.* Ann. Inst. Poincaré, Anal. Non Lin. 6 (1989), 73–93.

[6] W. Black, H. Howland, B. Howland. *A theorem about zig-zags between two circles.* Amer. Math. Monthly 81 (1974), 754–757.

[7] H. Bos, C. Kers, F. Oort, D. Raven. *Poncelet's closure theorem.* Expos. Math. **5** (1987), 289–364.

[8] J. Bracho, L. Montejano, D. Oliveros. *A classification theorem for Zindler carrousels.* J. Dynam. and Control Syst. 7 (2001), 367–384.

[9] R. Connelly, B. Csikos. *Classification of first order flexible regular bicycle polygons*, preprint.

[10] R. Connelly, I. Sabitov, A. Walz. *The bellows conjecture.* Beitrge Algebra Geom. 38 (1997), 1–10.

[11] B. Csikos. *On the rigidity of regular bicycle (n, k)-gons.* Contrib. Discrete Math. 2 (2007), 93–106.

[12] D. DeTurck, H. Gluck, D. Pomerleano, D. S. Vick. *The four vertex theorem and its converse.* Notices Amer. Math. Soc. **54** (2007), 192–207.

[13] F. Dogru, S. Tabachnikov. *On polygonal dual billiard in the hyperbolic plane.* Reg. Chaotic Dynam. 8 (2003), 67–82; also www.math.psu.edu/tabachni/prints/preprints.html

[14] F. Dogru, S. Tabachnikov. *Dual billiards*. Math. Intelligencer 27, No 4 (2005), 18–25; also www.math.psu.edu/tabachni/prints/preprints.html

[15] D. Finn. *Can a bicycle create a unicycle track?* College Math. J., September, 2002.

[16] R. Foote. *Geometry of the Prytz planimeter*. Rep. Math. Phys. **42** (1998), 249–271.

[17] D. Fuchs, S. Tabachnikov. *Self-dual polygons and self-dual curves*. Funct. Anal. & Other Math., in print; also arXiv:0707.1048 or www.math.psu.edu/tabachni/prints/preprints.html

[18] D. Genin, B. Khesin, S. Tabachnikov. *Geodesics on an ellipsoid in Minkowski space*. l'Enseign. Math., in print; also arXiv:0705.0188 or www.math.psu.edu/tabachni/prints/preprints.html

[19] D. Genin, S. Tabachnikov. *On configuration space of plane polygons, sub-Riemannian geometry and periodic orbits of outer billiards*. J. Modern Dynamics, 1 (2007), 155–173; also math.DG/0604388 or www.math.psu.edu/tabachni/prints/preprints.html

[20] M. Ghomi, S. Tabachnikov. *Totally skew embeddings of manifolds*. Math. Z., in print; also math.DG/0307044 or www.math.psu.edu/tabachni/prints/preprints.html

[21] Ph. Griffiths, J. Harris. *On Cayley's explicit solution to Poncelet's porism.* Enseign. Math. 24 (1978), no. 1-2, 31–40.

[22] E. Gutkin, S. Tabachnikov. *Complexity of piecewise convex transformations in two dimensions, with applications to polygonal billiards.* Moscow Math. J., 6 (2006), 673–701.

[23] J. Kantor, M. Maydanskiy. *Triangles gone wild.* MASS selecta, 277–288, Amer. Math. Soc., Providence, RI, 2003.

[24] B. Khesin, S. Tabachnikov. *Spaces of pseudo-Riemannian geodesics and pseudo-Euclidean billiards*. preprint math.DG/0608620 or www.math.psu.edu/tabachni/prints/preprints.html

[25] J. Lagarias, T. Richardson. *Convexity and the average curvature of plane curves.* Geom. Dedicata 67 (1997), 1–30.

[26] E. Larson. *The DNA inequality in nonconvex regions.* http://arxiv.org/abs/0801.1929

[27] M. Levi, S. Tabachnikov. *On bicycle tire tracks geometry, hatchet planimeter, Menzins conjecture and oscillation of unicycle tracks.* arXiv:0801.4396 or www.math.psu.edu/tabachni/prints/preprints.html

[28] P. Monsky. *A conjecture of Stein on plane dissections.* Math. Z. 205 (1990), 583–592.

[29] A. Nazarov, F. Petrov. *On S. L. Tabachnikov's conjecture.* Algebra and Analysis 19 (2007), 177–193.

[30] V. Ovsienko, S. Tabachnikov. *Projective differential geometry, old and new: from Schwarzian derivative to cohomology of diffeomorphism groups.* Camb. Univ. Press, 2005.

[31] S. Stein, S. Szabo. *Algebra and tiling.* Math. Assoc. Amer., 1994.

[32] S. Tabachnikov. *Commuting dual billiard maps.* Geom. Dedicata 53 (1994), 57–68.

[33] S. Tabachnikov. *Billiards.* Soc. Math. France, 1995; also www.math.psu.edu/tabachni/Books/books.html

[34] S. Tabachnikov. *A tale of a geometric inequality.* MASS Selecta, AMS, 2003, 257–262 or www.math.psu.edu/tabachni/prints/preprints.html

[35] S. Tabachnikov. *Geometry and billiards.* Amer. Math. Soc., 2005; also www.math.psu.edu/tabachni/Books/books.html

[36] S. Tabachnikov. *Tire track geometry: variations on a theme.* Israel J. Math. **151** (2006), 1–28; also math.DG/0405445 or www.math.psu.edu/tabachni/prints/preprints.html

[37] S. Tabachnikov. *A proof of Culter's theorem on the existence of periodic orbits in polygonal outer billiards.* Geom. Dedicata **129** (2007), 83–87; also arXiv:0706.1003 or www.math.psu.edu/tabachni/prints/preprints.html

[38] S. Tabachnikov. *Converse Sturm-Hurwitz-Kellogg theorem and related results*. J. Fixed Points Theor. **3** (2008), 121–130; also arXiv:0710.5902 or www.math.psu.edu/tabachni/prints/preprints.html

[39] S. Tabachnikov. *On algebraically integrable outer billiards.* Pacific J. Math. **235** (2008), 89–92; also arXiv:0708.0255 or www.math.psu.edu/tabachni/prints/preprints.html

[40] A. Tumanov, V. Zharnitsky. *Periodic orbits in outer billiard.* Int. Math. Res. Not. 2006, Art. ID 67089, 17 pp.; also math.DS/0608529.

[41] F. Wegner. *Floating bodies of equilibrium.* Stud. Appl. Math. **111** (2003), 167–183.

[42] F. Wegner. *Floating bodies of equilibrium. Explicit solution*. preprint physics/0603160.

[43] F. Wegner. *Floating bodies of equilibrium in 2D, the tire track problem and electrons in a parabolic magnetic field*. preprint physics/0701241.