Alexandrov's Conjecture: On the Intrinsic Diameter and Surface Area of Convex Surfaces

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Abstract

The following paper considers Alexandrov's conjecture, that the ratio of surface area to intrinsic diameter squared of a Riemannian 2-manifold with non-negative curvature has a maximum of $\frac{\pi}{2}$, for several classes of surfaces, including tetrahedra, pyramids with an arbitrary number of sides, degenerate convex surfaces, and general convex surfaces. We offer a new proof for the maximum of the ratio over tetrahedra (this is a previously known result), and new results for the maximum of this ratio over pyramids with an arbitrary number of faces and degenerate convex surfaces. We also develop proof techniques which may prove valuable in solving the conjecture for general convex surfaces.

1 Introduction

This paper seeks to investigate the maximum of the following function:

$$F(M) = \frac{Surface\ Area}{Intrinsic\ Diameter^2} \tag{1}$$

where M is a 2-manifold. The first class of manifolds we deal with will be polytopes, that is, convex polyhedra. We only consider convex surfaces because F can be made arbitrarily large over manifolds with negative Gaussian curvature. This function is clearly homothety invariant, so throughout the paper we may, without loss of generality, take the liberty of assigning an arbitrary side length to our polytopes. We will denote the geodesic distance between x and y (that is, the minimum distance between x and y, where distance is measured along the surface without entering the ambient space) by $g(x, y)$. Unless explicitly stated otherwise, diameter should be taken as intrinsic diameter, the maximum geodesic distance between any two points on the surface, or, more rigorously, $sup\{g(x, y)|x, y \in M\}$.

A.D. Alexandrov [2] conjectured in 1955 that F attains a maximum of $\frac{\pi}{2}$ over all Riemannian 2-manifolds. It can easily be seen that this value is attained by the doubly-covered disk, a degenerate convex surface formed by gluing two disks along their boundaries, formulating an equivalent conjecture that given the class of convex surfaces with diameter d, surface area is maximized for the doubly-covered disk.

The first result obtained was an upper bound of $F < \pi$ which was obtained from the Bishop volume comparison theorem. Regarding further upper bounds on F, we have two main results. As the reader will see, the second makes the first unnecessary, but we present both here to give a more complete history of the problem.

Theorem [Sakai [5], 1989] Let (S^2,g) be a Riemannian structure with nonnegative curvature on the 2-sphere. Then $F(S^2, g) < .985\pi$.

Theorem [Shioya [6], 1993] For any nonnegatively curved Riemannian metric g on the 2-sphere S^2, we have $F(S^2, g) \leq (\frac{5}{2}\sqrt{10} - 7)\pi < .906\pi$.

The other historical result we present here deals specifically with tetrahedra and will be discussed in detail in the next section.

Another aspect of the problem worth mentioning is the difficulty of actually determining the intrinsic diameter of a 2-manifold. A paper [Agarwal [1], 1997] developing an algorithm for finding the intrinsic diameter of a polytope using a star unfolding technique is, to our knowledge, the most current research in that area. However, it is fairly inefficient, running in $O(n^8 \log(n))$ time, and has not yet manifested into an actual program as far as we could determine.

You will see that most of the time, once the intrinsic diameter is determined, the problem usually comes to a solution using basic geometric techniques.

Figure 1: Tetrahedron unfolded into the plane of ABC.

2 Tetrahedra

Theorem [Makai [4], 1973] For any tetrahedron ABCD, $F(ABCD) \leq 3\sqrt{3}/4$. This holds with equality if and only if ABCD is a regular tetrahedron.

Here we give a different proof than Makai originally did in order to be able to generalize the proof of the tetrahedron case to pyramids later in the paper, but we begin by following the beginning of Makai's paper.

Proof. First we prove that for a regular tetrahedron, the intrinsic diameter comes between the apex and the center of the base triangle. Let our regular tetrahedron be called $ABCD$. Let O be the center of the face ABD, and let our edge length be 2. Then the geodesic diameter between O and C is $g(C, O) = 4\sqrt{3}/3$. Let P and Q be any points contained in the faces ABD and BDC, respectively. Then we cover the plane by rolling our tetrahedron repeatedly about its edges and labeling the points touched by A, B, C, and D accordingly. We now have a net of triangles covering the plane. We pick a face ABD and label P_0 where P would normally lie in our tetrahedron's face. Similarly, we label Q_i in each triangle BDC on the plane. If we now erase our labelings of A, B, C, and D on the plane, we are left with a net of the points Q_i with P_0 lying in one regular triangle formed by $Q_iQ_jQ_k$. This regular triangle has side length 4 and we have that $g(P, Q) \leq min(PQ_i, PQ_j, PQ_k) \leq 4\sqrt{3}/3$. This proves that for any P we choose, we can be at most $4\sqrt{3}/3$ away from Q. Since our surface area is $4\sqrt{3}$ we have that $F = 3\sqrt{3}/4$.

Now, we let $ABCD$ be an arbitrary tetrahedron. We unfold the faces DAC, DCB, and DBA about the edges AC, CB, and BA respectively into the plane of ABC (see Figure 1). Since we now have three D's, we let D_i be the vertex D that is opposite vertex i in our unfolding. We assume that our unfolding has points A, B, and C either on the boundary or in the interior of the triangle $D_AD_BD_C$. If A, B, and C happen to lie outside this triangle, we just relabel our vertices and unfold again around our new base ABC such that A, B, and C do not lie outside the triangle. We circumscribe the triangle $D_AD_BD_C$ with a circle Γ having center Q. Without loss of generality, we can assume that the radius of Γ is 1. For our proof, we must now prove that Q

Figure 2: Tetrahedron folded all the way into the plane of ABC.

lies inside the triangle ABC and depart from Makai's proof.

Assume for now that in the unfolding of our tetrahedron from above, the vertices A, B, and C lie in the segments D_BD_C, D_AD_C, D_AD_B respectively. Then, since A, B, and C necessarily lie on the midpoints of D_BD_C, D_AD_C, D_AD_B, proving that Q lies inside ABC is equivalent to proving that Q lies inside $D_AD_BD_C$. Proving that Q lies inside $D_AD_BD_C$ is equivalent to proving that the points D_A, D_B, D_C do not lie in any open half circle of γ. Without loss of generality we assume that $\angle D_AD_BD_C \geq \angle D_CD_AD_B \geq \angle D_BD_CD_A$. Thus, proving that D_A, D_B, D_C do not lie in any open half circle is equivalent to proving our claim: $\alpha \leq \pi/2$ where $\alpha \equiv \angle D_AD_BD_C$.

Proof of claim: Since A, B, and C lie on the midpoints of D_BD_C, D_AD_C, D_AD_B, our triangle $D_AD_BD_C$ is composed of four similar triangles, ABD_C, D_ABC, AD_BC, and ABC with $\alpha = \angle BD_AC = \angle AD_CB = \angle BAC = \angle ACD_B$. When triangles AD_BC and $ABDC$ are folded up, segments ADB and ADC must meet at some point. But if $\alpha > \pi/2$, then the closest these two sides can get is when both sides are folded all the way over until they are both again in the plane of ABC, and when they are folded like this, D_B and D_C can never meet (see Figure 2).

Recall that we assumed that the vertices A, B, and C lie in the segments D_BD_C, D_AD_C, D_AD_B respectively. Our argument still holds since by moving the vertex B inwards along the bisectors of angles $\angle D_AD_BD_C$ we can only increase our angle $\angle ABD_C$. Vertices A and C hold similarly. This concludes the proof of our claim.

The reader should notice that if $\alpha = \pi/2$ then D_A, D_B, D_C lie in a closed half circle. In this case, the only possible tetrahedron that can be constructed is the degenerate tetrahedron folded with triangles AD_BC and ABD_C folded flat on top of triangles D_ABC and ABC.

Now that we know that Q, the center of our circumcircle, lies in the interior of triangle ABC, we know that $g(D, Q) = 1$, and thus, that the intrinsic diameter of $ABCD$ is at least 1. If it happens to be larger, F can only decrease, so we do not bother trying to prove what the intrinsic diameter actually is right now.

For any unfolding of a tetrahedron that does not have the vertices A, B, and C lying in the segments D_BD_C, D_AD_C, D_AD_B respectively, if we unfold as above, we can clearly increase the surface area by moving A, B, and C until they do lie in these segments. Thus, we must now only maximize surface area for a tetrahedron that has A, B, and C in these segments. This is the same as maximizing the area of a triangle inscribed in our circle of radius 1. This maximum obviously comes when our triangle is an equilateral triangle, and this equilateral triangle folds up into our regular tetrahedron. Note that we cannot move A, B, or C outside the triangle $D_A D_B D_C$ because of our assumption above. Intuitively, this is because once we do, our intrinsic diameter jumps from being between some point in the interior of the base and the apex to being between some point on the boundary of the base and a point on the opposite face. Also, we know that if A, B, or C did lie outside the triangle, we could rename our vertices and unfold again to get A, B, and C inside the triangle. Therefore, if A, B, and C inside the triangle do not maximize F, A, B, and C outside the triangle do not either. This concludes the proof of our theorem. □

3 Pyramids I

Now we discuss pyramids. We define a pyramid to be a convex polytope with n faces. One of these faces is a base polygon with n-1 sides, and the other n-1 faces are triangles. Similarly to the tetrahedron, we try to unfold pyramids and maximize F. However, this time we use a little different type of unfolding. Instead of unfolding around a base, we use a star unfolding as in [1] and [3]. We explain later what the star unfolding is. First, we will state and prove our first results about pyramids.

Let P_n be the set of all pyramids having n faces such that n-1 of the faces are congruent isosceles triangles and the other face is a regular polygon having n-1 sides. Then we have the following three theorems:

Theorem 1. *If n=5 or n=7, then the maximum of F over* P_n *comes for a pyramid* ρ *such that* $H(\rho) = G(\rho)$.

Theorem 2. *If n=6 or* $n \geq 8$ *then the maximum of F over* P_n *comes from the pyramid with zero height.*

Theorem 3. *As n approaches infinity,* P_n *becomes the set of right cones with circular bases, and the maximum of F over this limit set comes when our cone has zero height (and therefore, is the doubly covered disk).*

Figure 3: Pyramid with φ and c labeled appropriately.

Before proving the theorems, we should define h, a, and b that were used in Theorem 1. For a pyramid $\rho \in P_n$, $H(\rho)$ is the length of the shortest path between the center of the base and the apex. $G(\rho)$ is the length between a vertex of the base and the farthest point from this vertex. One should notice immediately that for all $\rho \in P_n$, the intrinsic diameter is either $G(\rho)$ or $H(\rho)$. The proof of the theorems is largely calculational because we have very nice pyramids for which we can directly calculate F.

Proof of Theorems. The proof of the tetrahedron case above gives us a good idea of how to attack the problem, and the proof relies on the fact that three points define a circle. We consider for now only the case in which our pyramid is a member of P_n (and therefore unfolds about the base such that all instances of the apex lie in a common circle). Let the angle across from each of the two sides of equal length on each triangular face be φ. Let s be the length of the sides of the base polygon. For any $\rho \in P_n$, we have that $H(\rho) = \frac{s(\tan(\pi/2-\pi/(n-1))+\tan(\varphi))}{2}$. Let $G(\rho)$ come between two points x and y. Then, we must have that $a(x,y) = b(x,y)$ where $a(x,y)$ is the distance between x and y when measured by first going up a triangular face and $b(x,y)$ is the distance between x and y when measured by first going across the base. Otherwise, we can move x or y a little bit and increase G. Finally let $\alpha = \frac{n-1}{2}(\pi - 2\varphi)$. Now we split to two cases.

Case 1: n is odd. Then, define t as the length of the sides that are not length s on the triangles and d as the distance from the apex down the edge along this triangle to one of the endpoints of the segment realizing G. Define $\beta = \pi/2 - \pi/(n-1) + \varphi$. Let c be the distance from one corner of the base to the opposite corner, then $c = \frac{s}{\sin(\pi/(n-1))}$. We have the following explicit equations: $a = \sqrt{t^2 + d^2 - 2dt\cos\alpha}$ and $b = \sqrt{(t-d)^2 + c^2 - 2(t-d)c\cos\beta}$. Since $a = b$, we have that $d = \frac{2tc\cos\beta - c^2}{2t\cos\alpha - 2t + 2c\cos\beta}$.

Case 2: n is even. Then define t as the height of the isosceles triangles and d as the distance from the apex to one of the endpoints of the segment realizing G. Let z be the length of the sides of the isosceles triangles, then $z = \sqrt{t^2 + \frac{s^2}{4}}$. Let c be the distance between a corner of the base and the midpoint of the opposite edge, then $c = \frac{s(\sin(\pi/2-\pi/(2n-2)))}{2\sin(\pi/(2n-2))}$. We have the following explicit equations: $a = \sqrt{z^2 + d^2 - 2dz\cos\alpha}$ and $b = c + t - d$.

Since $a = b$, we have $d = \frac{c^2+t^2-z^2+2tc}{2c+2t-2z\cos\alpha}$.

Now we can easily compute $F(\varphi)$ for any value of φ. We have a simple one variable maximization problem with the condition that $\pi/2 - \pi/n \leq \varphi < \pi/2$. When n=5, we have that F has a maximum of 1.2799 at $\varphi = 1.1310$. This maximum comes precisely when G=H. When n=7, F has a maximum of 1.3405 at $\varphi = 1.2213$. Again here, we have that G=H. When n=6 or $n \geq 8$, F has a maximum when $\varphi = \pi/2 - \pi/n$, that is when ρ is the flat pyramid. One should note that although we assumed $n > 4$, when we use this same process with n=4, we get that F has a maximum of $1.2990 = \frac{3\sqrt{3}}{4}$ at $\varphi = 1.0472 = \pi/3$. This is precisely what we proved above. This concludes the proof of Theorems 1 and 2, so now we finish Theorem 3.

If one computes the maximum of F for arbitrarily large n, one gets arbitrarily close to $\pi/2$, but to complete the proof we first need to understand intuitively what happened above. If we start with an arbitrary pyramid $\rho \in P_n$ and set φ very close to $\pi/2$, we have a very tall pyramid. This gives us a diameter between the center of the bottom and the apex and a very small F. F increases as we decrease the height of ρ until the diameter switches to being between a corner of the base and somewhere on the opposite face or edge. This critical value is when G=H, and we may have a maximum here. We also may have a maximum when we decrease the height of ρ all the way to zero. The maximum always comes between these two critical heights, and as n approaches infinity, F is always larger for the pyramid of zero height than it is for the pyramid that has G=H. Clearly a regular polygon with infinitely many sides is a disk, and we get that the maximum of F comes from the doubly covered disk. This concludes the proof of Theorem 3. □

4 Pyramids II

We now have a pretty good idea about how F behaves for pyramids, so we move on to consider general pyramids via the star unfolding. The basic idea of the star unfolding of a pyramid P about a point x is that we cut along the shortest paths from x to each vertex. If there happen to be more than one distinct shortest paths, we just pick one of them arbitrarily to cut along. Then, the pyramid is unfolded into a plane according to these cuts. Since our geodesics cannot intersect except at a point (in which case the geodesic terminates), we have that the star unfolding is a connected polygon in the plane.

Figure 4: Line segments that include x are cuts.

Figure 5: Star unfolding about x that is in interior of base.

Figure 4 shows an unfolding of a pyramid into the plane of the base with point x in the interior of the base and point y at the apex of the pyramid. We star unfold about x by cutting along the dotted lines. Figure 5 shows the star unfolding of the pyramid about x.

Figure 6 shows an unfolding of a pyramid with x on the boundary of the base and y on an opposite edge. We do a star unfolding about x by cutting along the dotted lines and the solid line segments that contain x. Figure 7 shows this star unfolding.

Now we state a very important theorem about star unfolding.

Theorem [Aronov and O'Rourke [3], 1992] The star unfolding of any convex polytope does not self-overlap.

Without this theorem, the star unfolding would not be very useful to us now. An interested reader can see [1] for applications of the star unfolding, including an algorithm to find the intrinsic diameter of a polytope.

Theorem 4. *For any pyramid ρ with n faces, F is maximized for some $\rho \in P_n$*

Sketch of Proof: If ρ is in some sense a right pyramid (the line through the apex through the center of the base is orthogonal to the base), then all we have to do to increase F is stretch the shorter sides until they are all the same length as the longest side. This increases surface area drastically and barely changes diameter.

Now we consider the case that ρ is not a right pyramid. First, we use the star unfolding about a point x where points x and y are points that realize the diameter, and x is somewhere on the base, possibly on the boundary. Then, there are at least three equivalent shortest paths from x to y in the unfolding (notice that there are either n or n-1 instances of x in this unfolding). The existence of three unique ones is guaranteed because otherwise we would be

Figure 6: Line segments that include x are cuts.

Figure 7: Star unfolding about x that is on boundary of base.

able to move x slightly and increase diameter. Thus, we have a circle defined by three points with y at the center. Clearly all the other instances of x lie outside our circle because if they did not, then there would be a shorter path from x to y. Now we repeatedly deform this figure and increase surface area without increasing the diameter. We do this by replacing the x's that lie outside the circle with x's lying on the circle. This decreases surface area slightly, but then we maximize for the new pyramid given that all instances of x lie on the circle. This maximization adds more area than was originally lost because the instances of x lying outside the circle have to be fairly close to the circle–otherwise x and y would not realize our diameter. If our base is not regular, we can again increase surface area as we did above by stretching the shorter sides and leaving the x's fixed on the circle. The only tricky part of this proof seems to be guaranteeing that our diameter does not jump from being G to being H as they were defined above. Clearly, this is not a problem for most pyramids, only ones that are in a certain range of height (recall that above the pyramids with smaller height had G as the diameter and taller pyramids had H as the diameter). Since we're not taking short pyramids and turning them into tall ones or vice versa, we only have to worry about this certain range. One reassuring fact about this process is that even if those pyramids in this range do have diameters that jump, the difference in G and H is not large enough relative to the added surface area to decrease F. Thus, we have ended up with a $\rho \in P_n$ with larger F than the original ρ. $\square$

5 Degenerate Convex Surfaces

Given a closed, convex curve in the plane, its associated doubly-covered convex surface is formed by gluing two copies of the surface bounded by the curve together at equivalent points on their boundaries. We will refer to the two halves of the surface separated by the boundary as faces, and the projection of a point as simply the equivalent point on the interior of the other face.

We will proceed to prove that $F \leq \frac{\pi}{2}$ for doubly-covered convex surfaces, but first we need a preliminary result.

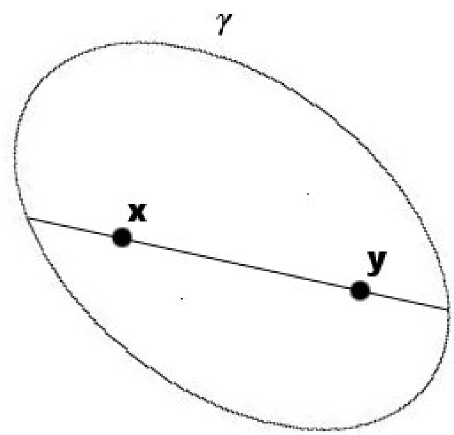

Figure 8: The chord containing two points on a degenerate convex surface

Lemma 5. *Given a doubly-covered convex surface,* ***S****, its intrinsic diameter is equal to the Euclidean diameter,* d_E*, of its boundary,* γ*.*

d_E is the distance between two points on γ through the ambient space contained by γ. In this case, the ambient space is **S** itself, ergo there exist at least two points, p_1,$p_2 \in$ **S** such that $g(p_1, p_2) = d_E$. It suffices to prove that all other points have a geodesic distance not greater than d_E.

Let x, y $\in$ **S**. If they both lie on the same face, including γ, then there is a chord of γ that contains x and y. The maximum length of a chord is given by the Euclidean diameter, d_E. Therefore, $g(x, y) \leq d_E$.

Suppose x, y are on opposite faces. Then there is a chord that contains x and the projection of y, and vice versa. Because the two faces are identical, the chords meet at the same points along γ, forming a closed curve of twice the length of the chords. Since the maximum distance of two points along a closed curve is one-half of the curve's length, and the maximum length of a chord in **S** is d_E, $g(x, y) \leq \frac{2d_E}{2}$, which is what we set out to prove.

Theorem 6. *$F \leq \pi/2$ for doubly-covered convex surfaces.*

Given the previous lemma, this statement is equivalent to proving that the disk is the convex, planar body containing the greatest area for any given Euclidean diameter.

This result is known as the isodiametric theorem, and was proven as early as the 1880's.

6 General Convex Surfaces

Any result proven over smooth, convex surfaces can, in effect, be extended to convex surfaces with polygonal boundaries, since any convex surface can be approximated arbitrarily closely by a smooth convex surface. One issue that arises in the transition from smooth to polygonal convex surfaces is the introduction of singularities in the form of vertices, but that can generally be worked around since length-minimizing geodesics do not intersect vertices.

In essence, this leads to an avenue of approach to Alexandrov's conjecture opposite to that presented by the first sections - instead of approximating smooth surfaces with polygonal ones, try to prove the conjecture for smooth surfaces and then approximate the polygonal ones. What follows are the germs of two proof techniques that may be useful in attacking the conjecture.

First, given the convex, smooth, compact, non-degenerate surface γ embedded in Euclidean 3-space, fix a point, p, in its interior. The choosing of p may or may not make a difference in the final proof, but it might be easier to place it halfway along a major or minor axis.

Define a function, f, giving the distance from p to the surface along a half-line in a given direction. Since this surface is non-degenerate, we'll need two angles to describe the half-line, as thus: $f(\theta_1, \theta_2) = d(p, \gamma(\theta_1, \theta_2))$, where d is standard Euclidean distance. This function inherits continuity and smoothness from those features of the surface.

The surface area local to $\gamma(\theta_1, \theta_2)$ can be approximated by the surface area of a Taylor polynomial in θ_1 and θ_2.

Several inequalities can be given, such as $f(\theta_1 + \pi, \theta_2) + f(\theta_1, \theta_2) \leq d$, and $f(\theta_1, \theta_2 + \pi) + f(\theta_1, \theta_2) \leq d$. However, these two inequalities do not completely describe the system. If a more complete description might be found, it may be possible to find a minimal upper bound for the surface area given a fixed diameter.

Secondly, given a strictly convex, smooth, non-degenerate surface **S**, there exists at least one minimum among the pairs of points whose normals coincide (referred to in some circles as the 2-periodic billiard trajectory of minimal length within the table bounded by the surface).

Using the normal line formed by one of these minimums as an axis, rotate

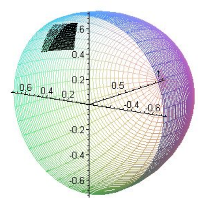

Figure 9: A local approximation of the surface area of an ellipsoid

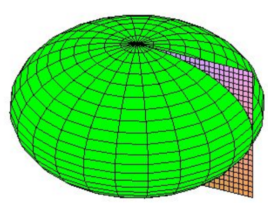

Figure 10: The foliating half-plane of an ellipsoid about the minor axis

a half-plane bounded by the axis around in a circle to foliate the surface with curves. Because the curves are the intersection of a plane with the surface, they are guaranteed to be planar curves, and because the axis was chosen so that the tangencies at the beginning and end of the curves are parallel to the foliating plane, they will all be legitimate functions in the plane.

The following diagram gives an example of one of the foliated curves at angle θ about the axis. Functions that will be defined on the curve, parameterized as $\gamma_\theta(x)$ with respect to the axis, are given as follows: $L_\theta(x)$, the length of the curve at distance x along the axis, and $\kappa_\theta(x)$, the planar curvature at point $\gamma_\theta(x)$. The length of the axis, r, is fixed for any family of foliated curves about a given axis.

By manipulating the above functions and their expressions in terms of x,

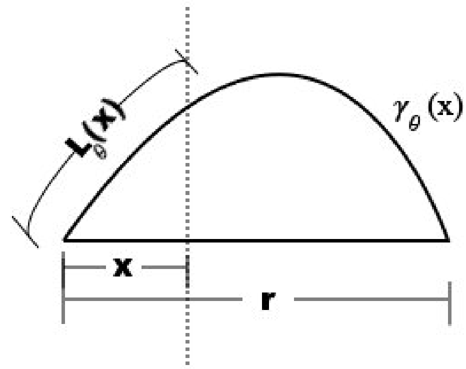

Figure 11: One of the curves foliating a manifold

given by $L_\theta(x) = \int_0^x (1 + (\gamma_\theta\prime(x))^2)^{\frac{1}{2}}$ and $\kappa_\theta(x) = \frac{\gamma_\theta\prime\prime(x)}{(1+(\gamma_\theta\prime(x))^2)^{\frac{3}{2}}}$, in addition to the formula for surface area of a surface of revolution of a parameterized curve, given by $\Delta\theta \int_0^x \gamma_\theta(x)(1 + (\gamma_\theta\prime(x))^2)^{\frac{1}{2}}$, where $\Delta\theta$ is the angle of revolution, we can determine a formula for the surface area of the entire surface area in terms of infinitesimal surfaces of revolution parameterized by $L_\theta(x)$ and $\kappa_\theta(x)$ as follows:

$$A_{\mathbf{S}} = \int_0^{2\pi} \int_0^r \frac{dL_\theta(x)}{dx} [\int_0^x \int_0^{x'} \kappa_\theta(x'')(\frac{dL_\theta(x'')}{dx''})^3 dx'' dx' + \gamma_\theta\prime(0) x' |_0^x] dx d\theta \quad (2)$$

This equation might not seem entirely helpful, and in addition, probably needs an extra factor of curvature in the inner-most integral to approximate the surface closely enough to make the integration valid.

To add to the problems, there's an undefined term, $\gamma_\theta\prime(0)$, multiplied by both x and zero on the far right-hand side (since γ_θ begins and ends with a vertical tangency, its derivative at 0 and r are undefined).

However, if some progress could be made in interpreting it, it might be possible to get at Alexandrov's conjecture. The doubly-covered disk has very distinct foliated curves (i.e., all of the same length, with one point of positive curvature and the rest having nearly zero curvature), so it follows that it would probably be a maximum of some sort of the above formula.

7 Conclusion

Since we are writing this paper during a brief summer research program, we have not been able to complete all of our endeavors. However, techniques similar to the techniques discussed in the sketch of Theorem 4 using the star unfolding might prove fruitful in proving Alexandrov's conjecture for all convex polytopes, and similarly for smooth convex surfaces in the last section.

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