THEN DNA INEQUALITY FOR A NON-CONVEX CELL

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Abstract.

1. Introduction

The DNA inequality is a comparison of two closed curves on the plane. The inequality states that the mean curvature of a convex closed curve Γ is less than or equal to the mean curvature of a closed, possibly self-intersecting curve, γ enclosed in it. The outer curve Γ is referred to as the cell, and γ is called the DNA for obvious reasons.

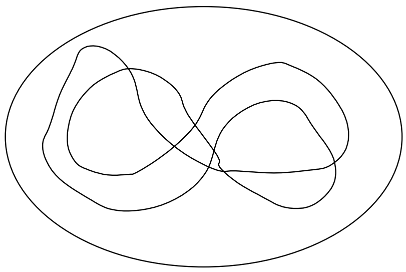

Figure 1. An example of Γ and γ for the DNA inequality

The mean curvature is defined for a curve γ as the total absolute curvature of γ divided by the length of γ. The total absolute curvature, denoted by $c(\gamma)$, is the integral over γ of the absolute curvature at each point of γ. Our notation for the mean curvature is $m(\gamma) = \frac{c(\gamma)}{l(\gamma)}$.

The inequality can be easily proven for Γ a circle. In his paper, Tabachnikov suggests four different methods to prove this statement [**?**]. However, the next step, for Γ a convex curve, the proof quickly becomes much more difficult. It was proven by Lagarias and Richardson in a quite complicated manner [**?**]. Their proof assumes a counter-example exists, then alters that counter-example in ways to reduce mean curvature to a convex shape. Since the DNA inequality can be easily shown for a convex curve γ inside convex curve Γ, we have a contradiction. Currently this is the only known proof for Γ as a convex cell.

In their paper on the convex case, Lagarias and Richardson conjecture that the DNA inequality holds for a certain non-convex cell[**?**]. Their particular conjecture is that the DNA inequality holds when the cell, Γ , is a unit square with a square

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of side length 2/3 or greater cut out of one corner, which shall here forth be called the L-shape.

Unfortunately, my many attempts to solve this problem went unfulfilled. I eventually decided to simplify the problem, in hopes that this lesser condition may lead to solving the more general question. I chose to simplify this question by imposing the structure of a unit square lattice on the curves. This structure requires that each segment of the curves Γ and γ travel in either a vertical or horizontal direction.

Since Γ is now one a unit square lattice it does not make sense to define our L-shape as a unit square with a square removed. This problem is easily fixed by scaling Γ by some constant b.

Now our problem can be stated as such:

Theorem 1.1. *For a Γ and γ on a unit square lattice, if Γ is the L-shape and γ is a curve inside of Γ then $m(\Gamma) \leq m(\gamma)$.*

2. The Rectangle on a Square Lattice

The first step to understanding the proof for the L-shape is understanding the DNA inequality for a rectangle on the square lattice. The inequality reads as follows; Assume Γ is a an $a \times b$ rectangle on a unit square lattice. If γ is a closed curve on the square lattice contained in Γ, then $m(\Gamma) \leq m(\gamma)$.

Currently two proofs are known for the case of the rectangular cell. The first is cases by length using induction.

Lemma 2.1. *if $(n-1)l(\Gamma) < l(\gamma) \leq nl(\Gamma)$, for $n \in \mathbb{N}$ then $c(\gamma) \geq 2\pi n$*

Proof: By Induction

For $n = 1$ the statement is obvious, because any closed curve has curvature 2π or greater.

for $n = 2$ if $c(\gamma) = 2\pi$ then length is maximized by the rectangle on the boundary so $l(\gamma) \leq 2(a+b) = l(\Gamma)$ if $c(\gamma) = 3\pi$ then length is maximized by with an L-shape where four of the edges are on the boundary so $l(\gamma) \leq 2(a+b) = l(\Gamma)$ Thus for $l(\gamma) > l(\Gamma) c(\gamma) \geq 4\pi$

Assume the statement is true for $n = k$. It must be shown that $c(\gamma) \geq 2(k+1)\pi$ for $(k)l(\Gamma) < l(\gamma) \leq (k+1)l(\Gamma)$

since $l(\gamma) > kl(\Gamma) > (k-1)l(\Gamma)$ we can assume that $c(\gamma) \geq 2k\pi$. if $c(\gamma) = 2k\pi$ then the length is maximized by k turns around the rectangle so $l(\gamma) \leq 2k(a+b) = kl(\Gamma)$. if c$(\gamma) = (2k+1)\pi$ then the length is maximized by k-1 turns around the rectangle and one L-shape, so $l(\gamma) \leq 2k(a+b) = kl(\Gamma)$. Thus $c(\gamma) \geq 2(k+1)\pi$.

□

Proposition 2.2. $m(\gamma) \geq m(\Gamma)$*, for every* $(n-1)l(\Gamma) < l(\gamma) \leq nl(\Gamma)$*, for* $n \in \mathbb{N}$

Proof: $m(\gamma) = \frac{c(\gamma)}{l(\gamma)} \geq \frac{2n\pi}{l(\gamma)} \geq \frac{2n\pi}{nl(\Gamma)} = \frac{2\pi}{l(\Gamma)} = m(\Gamma)$

This proposition shows that the DNA inequality holds for all lengths of γ

□

The second proof for the rectangle was offered by Sergei Tabachnikov and uses Integral Geometry[**?**]. Again the cell, Γ , is a $a \times b$ rectangle.

Proof: Let u =the number of horizontal (or vertical) segments in γ. Thus $c(\gamma) = u\pi$ since γ is contained by Γ the horizontal segments are bounded by a and the vertical segments are bounded by b. Thus the $l(\gamma)$ is maximized by $u(a+b)$.

$$m(\gamma) = \frac{c(\gamma)}{l(\gamma)} = \frac{u\pi}{l(\gamma)} \geq \frac{u\pi}{u(a+b)} = \frac{2\pi}{2(a+b)} = m(\Gamma)$$

□

3. The L-shape

Theorem 1.1 shall be proven by reducing the curve γ to a set of closed polygonal curves, which will be notated as Ω. Then it will be shown that every curve in Ω satisfies the DNA inequality for Γ.

Put the bottom left hand corner of Γ at the origin. Let

$$P = \{(0,0),(0,c),(0,b),(c,0),(b,0),(c,b),(c,c),(b,c)\}$$

as seen in the figure above. Let S be the set of segments connecting two points of P vertically or horizontally. Thus Ω is defined

$$\Omega = \{\gamma | \text{ every segment of } \gamma \in S\}$$

4. Reductions on γ

In this section we will consider reductions on the curve γ. In the language developed around the DNA inequality the term reduction means to alter a curve γ in such a way as to reduce the mean curvature of the curve. The specific reductions for this project also reduce the number of segments and increase the number of edges. A segment is a straight piece of γ such that no more than one of the endpoints are in P. An edge is a straight piece of γ such that both endpoints are in P. Note that a curve which consists only of edges is an element of Ω.

Assume γ is a curve on the square lattice in Γ, such that γ contains at least one point, $p \in P$. Now consider the first three segments of γ starting at p and label them x, y, and z. If the endpoints of x are both elements of P, then x is an edge, so we relabel y as x and z as y and z as the next segment of γ. Since we are restricted to a square lattice, these three segments are either horizontal, vertical, horizontal or vice versa. We will have two types of reductions, when x, and z are parallel and when x and z are anti-parallel.

Case 1: x and z are parallel.

Let the endpoint of z that is not shared with y be called t. Extend x to the vertex, $q \in P$, such that q is beyond t. Label the edge from p to q as $x\prime$. Now translate y to q. Label this segment $y\prime$, and its endpoint $r \notin P$. Finally connect r to t and label this segment $z\prime$.

Now we allow $\gamma\prime$ to be the curve γ with segments x, y, and z changed to $x\prime$, $y\prime$, and $z\prime$. It can easily be seen that: $c(\gamma) = c(\gamma\prime)$ and that $l(\gamma) \leq l(\gamma\prime)$, so that $m(\gamma) \geq m(\gamma\prime)$.

Case 2: x and z are anti-parallel.

Let the endpoint of z that is not shared with y be called t. Extend x to the next vertex, $q \in P$. Label the edge from p to q as $x\prime$. Now translate y to q. Label this segment $y\prime$, and its endpoint $r \notin P$. Finally connect r to t and label this segment $z\prime$.

Again, we allow $\gamma\prime$ to be the curve γ with x, y and z changed to $x\prime$, $y\prime$, and $z\prime$. In this case it is also easy to see that $c(\gamma) = c(\gamma\prime)$ and $l(\gamma) \leq l(\gamma\prime)$ so that $m(\gamma) \geq m(\gamma\prime)$.

After a successful reduction is made and $\gamma\prime$ is created, γ can be set to $\gamma\prime$. We make $y\prime$ the new x and $z\prime$ the new y, since $x\prime$ has been made into an edge. Now we can continue to reduce γ, until every straight piece of γ is an edge, and $\gamma \in \Omega$.

An astute reader may have noticed we assumed that γ contains an element of P. We can make this assumption because of the following argument. Assume γ is any curve on the square lattice inside of the L-shape. γ can be translated to the left until a segment of γ touches the left most segment of Γ. Label that segment x, and the two segments after it y and z. Extend x to the corner of Γ and label the new segment $x\prime$. Note that the corner of $\Gamma \in P$. Now translate y to the end of $x\prime$ and call it $y\prime$. Finally connect the end of $y\prime$ to the end of z and call it $z\prime$. Note that this reduction is almost identical to the reductions above. This reduction does reduce mean curvature, but it does not create a good edge. However, it send γ to an element of P, so that the next reduction will follow the reductions above.

Thus, any curve γ can be reduced to a curve in Ω.

5. The DNA inequality for a curve in Ω

Now that it has been shown that any curve γ can be reduced to a curve in the set Ω it remains to be shown that every curve in Ω has mean curvature greater than Γ.

Let $u =$ the number of horizontal (vertical) edges of γ. Thus $2u =$ the total number of edges, and $c(\gamma) = u\pi$.

Remember we are trying to show that

$$m(\gamma) = \tfrac{c(\gamma)}{l(\gamma)} \geq \tfrac{c(\Gamma)}{l(\Gamma)} = m(\Gamma)$$

By replacing the inequality with the definitions above we have

$$\tfrac{u\pi}{l(\gamma)} \geq \tfrac{3\pi}{4b}$$

and this can be multiplied to

$$l(\gamma) \leq \tfrac{4}{3}bu$$

Since $\gamma \in \Omega$, γ can only have edges of 4 lengths. These lengths are a, b, c and 0, as seen in figure ???. Let $A =$ the number of a's, $B =$ the number of b's, $C =$ the number of c's, and $D =$ the number of 0's. Since every segment is one of these four lengths we can say

$$2u = A + B + C + D \text{ and } l(\gamma) = Aa + Bb + Cc$$

Now the inequality we are trying to show can be written as

$$Aa + Bb + Cc \leq \tfrac{4}{3}(\tfrac{A+B+C}{2})b$$

Analysis of this inequality shows that

$$\begin{gathered} Aa + Bb + Cc = Ab - Ac + Bb + Cc = (A+B)b + (C-A)c \\ \leq (A+B)b + \tfrac{C-A}{3}b = (\tfrac{2}{3}A + B + \tfrac{1}{3}C)b \end{gathered}$$

The first equality comes from the fact that $a+c=b$, and the inequality comes from $c \leq \frac{1}{3}b$.

Now our inequality can be written as

$$\tfrac{2}{3}A + B + \tfrac{1}{3}C)b \leq \tfrac{4}{3}(\tfrac{A+B+C}{2})$$

Which can be simplified to

$$2A + 3B + C \leq 2A + 2B + 2C$$

and again to

$$B \leq C + 2D$$

So, now the DNA inequality has been reduced to the inequality $B \leq C + 2D$. First assume that γ makes no turns of π, this means there are no segments of length 0 and $D = 0$. Now all we have to show is $B \leq C$.

Let $S =$ the sequence of edges of γ labeled by length. Since γ is closed the first k entries of the sequence are the only unique entries, S repeats these k entries infinitely. Also, any elements of S can be assumed to be the first entry. Note that it is impossible to have more than two b's consecutively, and that there are only two ways to have two b's, as seen in Figure ???.

If $bb \notin S$, then each b is followed by at least one c. This can be shown by brute force in Figure ???. This fact easily shows that $B \leq C$.

if $bb \in S$. Assume bb begins S. Thus

$$S = bb \ldots b \ldots b \ldots bb \ldots$$

where there are any integer number of single b's between two bb's.

It was established in the previous case that there is at least one c between any two single b's, which means S above can be rewritten to

$$S = bb \ldots c \ldots b \ldots c \ldots bb \ldots$$

However, only one c between each b is insufficient to show that $B \leq C$. We need at least one more c to have the inequality. The argument follows by contradiction.

Assume there is only one c between the b's. Then γ can only traverse the two c's of Γ in the directions in Figure ???.

The proof of the above claim also follows by brute force. There are only two possible paths that γ can take after a bb that have only one c between two b's, which can be seen in Figure ???. These paths both end in the same place the start up to symmetry.

Thus it is impossible for there to be exactly one c between two b's and for S to have a second bb.

The final case is when we allow turns of π, so $d \geq 1$. Now there must a c or a d between any two b's. However, since d count twice it easily follows that $B \leq C+2D$, even when $bb \in S$.

6. Closing Remarks and Continued Research

I consider this work to be only a partial result. I believe the conjecture that any curve inside the L-shape will hold the DNA inequality. I believe the work I presented here can be used to generalize to any curve.

The part of my proof which is most heavily dependent on the square lattice structure is the reductions presented in section 3. If these reductions could be replaced by reductions that hold for any smooth or polygonal curve, then the DNA inequality will hold. The reductions used by Lagarias and Richardson for the convex case may also prove to be useful in developing the required reductions for the L-shape.

Finally, the work on the DNA inequality in the plane could be extended to other non-convex shapes. What makes the L-shape so special to have this characteristic? What other shapes could possibly have this characteristic? What are the necessary and sufficient conditions for the DNA inequality to hold for a non-convex cell?

There are many ways this question could be generalized for this research to continue.

References

[1] Lagarias and Richardson :
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