Lower Bounds on Mean Curvature of Closed Curves Contained in Convex Boundaries

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August 13, 2004

Abstract

We investigate a geometric inequality that states that in $\mathbb{R}^2$, the mean curvature of a closed curve γ contained inside of a convex closed curve Γ is not less than the mean curvature of the latter curve. It was first proved by Lagarias and Richardson in [5]. We attempted, but failed, to find a shorter and easier proof using integral geometry. However, we find a new proof of the inequality in the case when Γ is of constant width. We also prove the inequality in this case for curves on the sphere, and we show that the inequality does not hold in the hyperbolic plane.

1 Introduction

Let γ be a curve in $\mathbb{R}^2$ with length $l(\gamma)$. Its *total absolute curvature* $c(\gamma)$ is given by the integral of the absolute value of its curvature with respect to the arclength parameter. The *mean curvature* $m(\gamma)$ is total absolute curvature divided by length.

Consider a closed convex curve Γ and a closed, possibly self-intersecting, curve γ contained inside of Γ. By *convex*, we mean that the geodesic between any two points inside of Γ is completely contained inside of Γ. The same definition applies in spherical and hyperbolic geometries. We refer to Γ as the cell and γ as the DNA, and we have the following geometric inequality.

Theorem 1 (DNA inequality). *The mean curvature of the DNA curve γ is greater than or equal to the mean curvature of the convex cell curve Γ.*

Tabachnikov [6] conjectures that this inequality is an equality if and only if γ coincides with Γ, perhaps traversed more than once. These claims are easy to state, but apparently difficult to prove. Lagarias and Richardson [5] found a proof of Theorem 1, but their proof is rather long and involved, and it does not imply the conjecture. In this paper, we will explore some approaches to the inequality in hopes of finding a shorter and easier proof. We also look at the DNA inequality in spherical and hyperbolic geometry, and in the square lattice. The inequality is likely true in spherical geometry; however, we show that it is false in general in the hyperbolic plane.

It turns out that when Γ is a circle, the DNA inequality is easy to prove. Fáry [3] proved the inequality for this case in 1950 with integral geometry; Tabachnikov provides 3 more proofs using very different methods in [1]. We succeed in finding yet more proofs for the inequality in this case and in a few other special cases.

Sometimes it is more convenient to view the DNA γ as a smooth curve, and sometimes it is more convenient to view it as a polygonal line approximation. We will move from one view to the other freely. In the spherical and hyperbolic situations, we do the same, except instead of approximating by lines we approximate by the respective geodesics.

2 Integral geometry in the plane

Let N be the space of oriented lines in the plane, and choose an origin and a reference direction. An oriented line is uniquely determined by the angle θ it makes with the reference direction and its signed distance p from the origin; we denote such a line by $r(p, \theta)$. Note that p can be any real number and that θ ranges from 0 to 2π, so N is topologically equivalent to the infinite cylinder $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$.

Now consider a curve γ with a given orientation in the plane. Define $n_\gamma(p, \theta)$ to be the number of intersections that the line $r(p, \theta)$ makes with γ. Also define $k_\gamma(\theta)$ to be the number of times the oriented tangent of γ is in the direction θ. Then we have the following useful formulas.

Theorem 2.1 (Crofton formula). *The length of any curve γ in the plane is given by*

$$\frac{1}{4}\iint_N n_\gamma(p, \theta) dp d\theta.$$

Proof. The curve γ can be approximated with a polygonal line, and it suffices to prove the formula for polygonal lines. Let γ_0 be a line segment of unit length, and let

$$C = \iint_N n_{\gamma_0}(p, \theta) dp d\theta.$$

C only depends on the length of γ_0 because the space of oriented lines is invariant under isometry. By additivity of the integral, it follows that for any γ, we have

$$C \cdot l(\gamma) = \iint_N n_\gamma(p, \theta) dp d\theta.$$

To find the value of C, we will compute the integral for a unit circle γ centered at the origin. In this case, we have $l(\gamma) = 2\pi$ and

$$\iint_N n_\gamma(p, \theta) dp d\theta = \int_0^{2\pi} \int_{-1}^{1} 2 dp d\theta = 8\pi.$$

It follows that $C = 4$. $\square$

Theorem 2.2. *The total absolute curvature of any curve γ in the plane is given by*

$$\int_0^{2\pi} k_\gamma(\theta) d\theta.$$

Proof. Let γ_0 be an arc of a circle with radius r of central angle θ_0. We see that $c(\gamma) = \theta_0$. Clearly $k_\gamma(\theta) = 1$ as θ goes through the arc, and it is 0 for all other θ, so

$$\int_0^{2\pi} k_\gamma(\theta) d\theta = \theta_0 = c(\gamma).$$

Therefore the formula holds for γ_0 and the rest follows by additivity. $\square$

We now reproduce Fáry's proof of the DNA inequality in the case when the cell Γ is the unit circle. Note that $c(\Gamma) = 2\pi$, because the curvature of any convex closed curve is 2π. Thus $m(\Gamma) = 1$, and the desired inequality becomes $l(\gamma) \leq c(\gamma)$.

Theorem 2.3. *The DNA inequality holds when the cell* Γ *is a circle.*

Proof. Let γ be a smooth curve contained inside the unit circle Γ centered at the origin. It is clear that for all p, θ, we have $n_\gamma(p, \theta) \leq k_\gamma(\theta) + k_\gamma(\theta + \pi)$. It follows that

$$\begin{aligned} l(\gamma) &= \frac{1}{4} \iint_N n_\gamma(p, \theta) dp d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \int_{-1}^{1} n_\gamma(p, \theta) dp d\theta \\ &\leq \frac{1}{4} \int_0^{2\pi} 2(k_\gamma(\theta) + k_\gamma(\theta + \pi)) d\theta \\ &= \int_0^{2\pi} k_\gamma(\theta) d\theta \\ &= c(\gamma), \end{aligned}$$

which is the desired inequality. $\square$

Corollary. *The DNA inequality holds when the cell is a convex curve of constant width.*

Proof. $\square$

For any curve γ in $\mathbb{R}^2$, we define can a corresponding *dual curve* γ^* consisting of points (p, θ) in $N = \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ such that $r(p, \theta)$ is tangent to γ.

3 Dual Mapping from the Plane

The dual map is a map from smooth curves in the plane to curves on the cylinder. Pick an origin O, and a reference direction in the plane and let γ be a smooth curve in the plane. Each point of γ has a tangent line uniquely given by its angle with the reference direction and its distance from the origin. The dual curve to γ, γ^* is the image γ under the dual map. Lines are given parameters in $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ which is an infinitely long cylinder, so γ^* is a curve on the cylinder.It is clear that the dual of a smooth curve is closed since the tangent line is continuously changing everywhere.

We can use dual curves to gain some more geometric intuition about the proof by integral geometry of the DNA inequality in a circle. Let γ be a smooth curve contained inside Γ a unit circle. The proof by integral geometry used the n_γ function, the k_γ function and implicitly used the n_Γ function. These are all functions of lines in the plane, therefore can be viewed as functions on the cylinder. The support of the n functions will define areas and the support of the k functions will define curves. Also, the n functions are locally constant, and a change in the value of n, i.e. a change in the number of times lines intersect the curve, is due to the lines crossing a tangent to the curve which corresponds in the dual case to points passing the dual curve.

Fix a θ, and move up the cylinder starting at $-\infty$. The value of n_γ will be 0 until the line defined by (p, θ) intersects γ. Since γ is smooth, this happens precisely at the lowest tangent line to γ with direction θ or $\theta + \pi$. Every subsequent change in $n_(\gamma)$ will be due to the line crossing a tangent to γ, i.e. a point of γ^* or $-\gamma^*$. Because of this we can view the graph of the n_γ function as the area enclosed by γ^* and $-\gamma^*$.

The curvature formula can be interpreted as follows. $c(\gamma) = \int_0^{2\pi} k_\gamma(\theta) d\theta$, where $k(\theta)$ is the number of tangent lines to γ of direction θ, i.e. how many times γ^* attains the angle θ. Visually, the curvature of γ corresponds to the total radial movement of γ^*.

Pick the origin O to be the center of Γ. Now Γ^* and $-\Gamma^*$ become lines defined by $p = 1$ and $p = -1$ respectively, since Γ is a unit circle. If we let $A(\gamma)$ denote the area comtained between γ^* and $-\gamma^*$, and likewise for Γ, the DNA inequality becomes $\frac{l(\Gamma^*)}{A(\Gamma)} \leq \frac{l(\gamma^*)}{A(\gamma)}$.

4 Spherical geometry

Let γ be a closed curve on the unit sphere contained inside a convex closed curve Γ, also on the sphere. Note that a convex closed curve Γ divides the sphere into two regions. We refer to the smaller of the two regions.) We wish to show that the spherical mean curvature (total absolute spherical curvature per spherical length) of γ is greater than that of Γ. First we shall compute the spherical mean curvature of the circle Γ.

The spherical length of Γ on the sphere is the usual length when regarded in $\mathbb{R}^3$. Let the spherical radius of Γ be ρ. The radius of a circle on the surface of the unit sphere is the same as the angle made between the line through the center of the circle to the center of the sphere and any line connecting a point on the circle to the center of the sphere. To find the radius of the sphere in $\mathbb{R}^3$, we simply take $\sin(\rho)$. This makes sense because as we look at circles contained in smaller and smaller regions of the sphere, the region will look more and more like the plane, and $\sin(\rho) \approx \rho$ for small values of ρ. Also, a circle of spherical radius $\frac{\pi}{2}$ on the sphere is a great circle whose radius is the same as that of the sphere, and $\sin(\frac{\pi}{2}) = 1$. The length of Γ is therefore $2\pi \sin(\rho)$.

There are two ways of computing the spherical curvature of Γ, one by calculus and one by dual mapping and integral geometry. First we do the calculus method. The spherical curvature at a point of a curve is the curvature of the curve when projected onto the plane tangent to the sphere at that point. It is easy to see that the projection of the circle will look the same when projected onto the tangent plane at any of its points. Let Γ have spherical radius ρ. The projection will be an ellipse of width $2\sin(\rho)$ and height $2\sin(\rho) cos(\rho)$.

Figure 1: Dual Map

Therefore we can parameterize this projection of Γ, which we call Γ_p, by

$$\Gamma_p(t) = (\sin(\rho)\cos(t), \sin(\rho)\cos(\rho)\sin(t)).$$

Thus

$$\begin{aligned}
\Gamma_p'(t) &= (-\sin(\rho)\sin(t), \sin(\rho)\cos(\rho)\cos(t)), \\
\Gamma_p''(t) &= (-\sin(\rho)\cos(t), -\sin(\rho)\cos(\rho)\sin(t))
\end{aligned}$$

and

$$\begin{aligned}
\kappa(t) &= \frac{\|\Gamma_p'(t) \times \Gamma_p''(t)\|}{|\Gamma_p'(t)|^3} \\
&= \frac{\|\sin^2(\rho)\cos(\rho)\sin^2(t) + \sin^2(\rho)\cos(\rho)\cos^2(t)\|}{\|\sqrt{\sin^2(\rho) sin^2(t) + \sin^2(\rho)\cos^2(t)}\|^3}
\end{aligned}$$

But we are only interested in the curvature at the point where we took the projection, which by this parameterization is at $t = \frac{3\pi}{2}$. So we plug this in and find that curvature at a point of the circle is $\frac{\sin^2(\rho)\cos(\rho)}{\sin^3(\rho)} = \cot(\rho)$. Since the projection is the same at each point of the circle, so must the curvature; and the total spherical curvature is the length times the pointwise curvature, i.e. $c(\Gamma) = 2\pi\sin(\rho)\cot(\rho) = 2\pi\cos(\rho)$. Finally, $m(\Gamma) = \frac{c(\Gamma)}{l(\Gamma)} = \cot(\rho)$.

The other way of computing the curvature of the circle Γ is by integral geometry and dual mappings on the sphere. Like in the plane, we can define dual curves to curves on the sphere. The dual C^* of a curve C is the curve consisting of the north poles of the oriented great circles tangent to C. If C has a corner, then the dual traverses the geodesic (smaller arc of a great circle) connecting the north poles of the great circles which are right and left tangent to C at the corner. Consider a polygon P on the sphere. Its dual will attain only one point for each arc of P, and will traverse one arc for each corner of P. It is easy to see that since we are on the unit sphere, the angle between two great circles is equal to the geodesic distance between their north poles. Thus the curvature of P becomes the length of P^*. Since we can approximate any curve C on the sphere by polygonal lines, this holds for arbitrary curves on the sphere.

There is a Crofton formula on the sphere which is the exact analog of that in the plane. Let G_x denote the oriented great circle on the sphere whose north pole is at the point x in the sphere S. Given any curve γ, we can define for all $x \in S$ the function $n_\gamma(x)$ which is equal to the number of intersection points of G_x and C. Then we have $l(\gamma) = \frac{1}{4}\int_{x \in S} n_\gamma(x)dx$, where dx is the usual measure on the sphere (which is $\sin(\phi)d\phi d\theta$).

Now to find the curvature of a circle Γ, we need to compute the length of its dual, Γ^*. Let ρ be the spherical radius of Γ. This is also the angle between the center and edge of the circle from the center of the sphere. The radius of a circle is always perpendicular to the circle, so to find the north pole of the tangent great circle to Γ at a point p, we travel $\frac{\pi}{2}$ along the great circle containing the radial line from the center of Γ to p. Depending on the orientation of the circle, we will either go back across the Γ or continue in the direction of p away from the center of Γ. In one case we will end up at a point $\frac{\pi}{2} - \rho$ away from the center of Γ, and in the other we will end up the same distance away from the point antipodal to the center of Γ. Thus, Γ^* is a circle of radius $\frac{\pi}{2} - \rho$, whose length we already know to be $2\pi\sin(\frac{\pi}{2} - \rho) = 2\pi\cos(\rho)$, confirming our previous computation.

Theorem 5. *Let Γ be a circle on the sphere of spherical radius ρ. Then for any closed curve γ on the sphere contained inside of Γ, the DNA inequality holds: $m(\gamma) \geq m(\Gamma)$. Also, equality holds if and only if γ coincides with Γ, possibly traversed multiple times.*

We have two proofs of this theorem. One proof is in the style of the "proof by rolling" in the plane case which can be found in [6], and the other uses dual mappings and integral geometry.

Figure 2: Unrolling the Curve

Proof 1. Let γ be a closed curve inside the circle Γ centered at O with spherical radius ρ on the unit sphere. We approximate γ by arcs of great circles. We need to show that $m(\gamma) \geq m(\Gamma) = \cot(\rho)$. Let the vertices of γ be $v_1, \ldots, v_n$, and place another vertex v_0 anywhere on the segment $\overline{v_1 v_n}$.

We wish to "unroll" γ onto a great circle and trace the path of O through this unrolling. To do this, fix the segment $\overline{v_0 v_1}$ and rotate everything else on the sphere about the axis of the sphere with north pole v_1 until v_0, v_1 and v_2 lie on the same great circle. The direction of the rotation is the one which aligns the vertices with the least amount of rotation angle. For the next step we fix the segments $\overline{v_0 v_1}$ and $\overline{v_1 v_2}$ and again do a similar rotation about the axis with north pole v_2. Repeat this process n times, that is, until γ lies entirely on a great circle. The unrolled curve starts and ends with v_0. It is easy to see that at the i^{th} step we rotate through an angle of exactly α_i, which is the exterior angle of γ at the vertex v_i that we rotated about.

During the i^{th} turn, O travelled a distance of $\alpha_i \sin(\rho_i)$, where ρ_i is the spherical distance from v_i to O. Therefore the total distance travelled by O is $\sum_{i=1}^{n} \alpha_i \sin(\rho_i)$. We know $0 \leq \rho_i \leq \rho \leq \frac{\pi}{2}$, and $\sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, therefore $\sum_{i=1}^{n} \alpha_i \sin(\rho_i) \leq \sum_{i=1}^{n} \alpha_i \sin(\rho) = c(\gamma)\sin(\rho)$.

Figure 3: Boundary for O

We want to compare the total distance that O travelled to the least possible distance that it could have travelled during the unrolling. Since O cannot be farther than ρ from any vertex of γ at any moment of the unrolling, the path of O must be completely contained inside of the strip shown in Figure 3. Note that the angles at each of the corners of the strip are right angles and that the left and right boundaries of the strip are arcs of great circles. Also, the strip may wrap several times around the sphere, but this is not reflected in Figure 3 for simplicity. O starts on some point of the left vertical boundary and ends on the identical point on the right vertical boundary, which we denote by O'. This is because the leftmost and rightmost points on the unrolled γ are both v_0, which was by construction located on the segment $\overline{v_1 v_n}$. Therefore O and O' are in the same position with respect to the left and right ends of the unrolled curve γ, respectively.

We wish to find the length of the shortest path between any point on the left boundary and the identical point on the right boundary that is contained in the strip. It is obvious that crossing the great circle on which γ lies will not minimize length, so it is enough to look at the top half of the strip.

Figure 4: Shape in Question

Our claim is that the shortest path is the one directly along the top piece of the boundary. Suppose that the shortest path does not travel directly along the top piece of the strip. Call this path P. If it does not touch the strip at all, then clearly we can shift O, O', and P upwards until it touches the top boundary of the strip. This shifted path is clearly shorter. Therefore it must touch the top boundary of the strip at at least one point. Let p be the leftmost or the rightmost such point. Consider some point q on P before or after p that is not touching the top boundary. We can make the distance between q and p as small as we like, because the path P is continuous and because of our choice of p. If the segment of P between q and p is not the arc of a great circle, then the path is clearly not the shortest possible. So we may assume that this segment is an arc of a great circle. Let r be the point on the top of the strip closest to q, and then let a be the distance from q to r, let b be the distance between r and p along the top of the strip, and let c be the geodesic distance between q and p. Since we may choose q to be arbitrarily close to p, and since the geometry of the sphere is locally Euclidean, we see that for some choice of q the triangle inequality $a + b \leq c$ will be satisfied.

Repeating this argument as many times as necessary, we see that the shortest path between O and O' contained in the strip is the path along the top boundary. The length of the top boundary is $l(\gamma)\cos(\rho)$. Hence $l(\gamma)\cos(\rho) \leq c(\gamma)\sin(\rho) \Rightarrow m(\gamma) \geq \cot(\rho)$. □

Proof 2. The second proof is by dual mapping and integral geometry on the sphere. The first important fact to take note of is that when we take the dual curves of γ and Γ, the "containment" reverses, i.e. γ^* is completely outside of Γ^*. We have already seen that the length of γ^* is the curvature of γ. By the

Figure 5: Curves and Duals

spherical Crofton formula, we have

$$l(\gamma) = \frac{1}{4}\int_{x\in S} n_\gamma(x)dx = \frac{1}{4}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n_\gamma(\theta,\phi)\cos(\phi)d\phi d\theta,$$

where we pick the great cirle $\phi = 0$ to be concentric with Γ. It is easy to see that a great circle intersects Γ if and only if its north pole lies in the strip between Γ^* and $-\Gamma^*$. This is because if we consider a family of great circles $G(\theta, \phi)$ with θ fixed, $n_\Gamma(\theta,\phi)$ also varies between its minimum value of 0 and its maximum value of 2. In particular there will be two points at which $n_\Gamma(\theta,\phi) = 1$, i.e. some great circle is tangent to Γ. These great circles make up points of Γ^* and $-\Gamma^*$. Therefore the boundry of the support of $n_\Gamma(\theta,\phi)$ is Γ^* and $-\Gamma^*$, and the support of the function is the strip between the curves. If a great circle intersects γ, then since $\gamma \subset \Gamma$ the great circle must also intersect Γ. Therefore the support of $n_\gamma(\theta,\phi)$ is contained in the support of $n_\Gamma(\theta,\phi)$, i.e. if γ is contained in Γ a circle of spherical radius ρ, then γ^* and $-\gamma^*$ lie in a strip of width 2ρ around the great circle concentric with Γ. This is the same region bounded by Γ^* and $-\Gamma^*$. Now for the proof.

$$\begin{aligned}
l(\gamma) &= \frac{1}{4}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n_\gamma(\theta,\phi)\cos(\phi)d\phi d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_{-\rho}^{\rho} n_\gamma(\theta,\phi)\cos(\phi)d\phi d\theta
\end{aligned}$$

$$\int_{-\rho}^{\rho} n_\gamma(\theta,\phi)\cos(\phi)d\phi \le \max_\phi\{n_\gamma(\theta,\phi)\}\int_{-\rho}^{\rho}\cos(\phi)d\phi$$
$$= 2\max_\phi\{n_\gamma(\theta,\phi)\}\sin(\phi)$$

We also know that if we fix θ and let ϕ vary, every change in the value of n_γ is the result of crossing γ^* or $-\gamma^*$. On the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$, n_γ starts and ends with the value 0. Therefore there must be as many steps up as down, i.e. $\max_\phi\{n_\gamma(\phi,\theta)\} \le (\#\ \ \gamma^*,\ -\gamma^* = \theta)$.

Hence

$\frac{1}{4}\int_0^{2\pi}\int_{-\rho}^{\rho} n_\gamma(\phi,\theta)\cos\phi d\phi d\theta \le \frac{1}{4}\int_0^{2\pi} 2(\#\gamma^*,\ -\gamma^* = \theta)\sin\rho$

As we showed in the first proof, the shortest distance around a strip is along the top boundry, therefore we know

$c(\gamma) = l(\gamma^*) \ge \int_0^{2\pi}\cos\rho(\#\gamma^* = \theta) \Rightarrow \int_0^{2\pi}(\#\gamma^* = \theta) \le \frac{c(\gamma)}{\cos\rho}$

Combining our calculations we get

$l(\gamma) \le \frac{1}{4}\int_0^{2\pi} 2(\#\gamma^*, -\gamma^* = \theta)\sin\rho = \sin\rho\int_0^{2\pi}(\#\gamma^* = \theta) \le c(\gamma)\frac{\sin\rho}{\cos\rho}$

$$\Rightarrow \cot\rho \le m(\gamma)$$

□

5 Hyperbolic geometry

We shall consider the upper-half plane model of the hyperbolic plane, which is given by the set $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

In this model, geodesics are straight vertical lines and semicircles orthogonal to the real line. A unique geodesic passes through any pair of distinct points in $\mathbb{H}^2$. Furthermore, the notion of hyperbolic angle measure in $\mathbb{H}^2$ coincides with the Euclidean one.

A curve γ in the hyperbolic plane may be given by $\gamma(t) = x(t) + iy(t)$ with $t \in [0,1]$ for some real-valued functions $x(t)$ and $y(t)$. We define the hyperbolic length of γ to be

$$l(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)}dt. \tag{1}$$

Naturally, the hyperbolic distance $\rho(z,w)$ between two points $z, w \in \mathbb{H}^2$ is defined as the length of their geodesic arc. A routine calculation gives the formula

$$\rho(z,w) = \log\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}.$$

In the Euclidean plane, we approximated curves by polygonal lines, and on the sphere, we approximated curves by arcs of great circles. The total absolute curvature $c(\gamma)$ of an approximation curve γ in these cases was given by the sum of the exterior angles at each of the vertices of γ. The same principles carry over to the hyperbolic plane, except with hyperbolic geodesics instead of Euclidean or spherical ones, of course.

Figure 6: Figure 1

Example. Consider the horizontal line of Euclidean height $h > 0$ and Euclidean length 1 given by $\gamma(t) = t + ih$. It is clear that the hyperbolic length is $l(\gamma) = 1/h$. Note that the endpoints of γ are ih and $1 + ih$. Let γ_n be the curve consisting of the geodesics connecting the points $\frac{j}{n} + ih$ and $\frac{j+1}{n} + ih$ for $j = 0, 1, \ldots, n-1$. The exterior angle at each vertex of γ_n is given by $\beta = 2\alpha = 2 \arctan \frac{1}{2hn}$ (see Figure ??), and so $c(\gamma_n) = 2(n-1) \arctan \frac{1}{2hn}$. Since $\arctan x \approx x$ as $x \to 0$, it follows that $\lim_{n\to\infty} c(\gamma_n) = 1/h$. Clearly, $\lim_{n\to\infty} \gamma_n = \gamma$, so $c(\gamma) = 1/h$, and the mean curvature $m(\gamma)$ is exactly 1.

It easily follows that any other horizontal line of Euclidean height $h > 0$ and Euclidean length $k > 0$ has hyperbolic length k/h, curvature k/h, and mean curvature 1.

Figure 7: Figure 2

We now show that the DNA inequality does not hold in the hyperbolic plane. Consider the closed curve Γ in Figure ??. The vertices x and y are connected by a geodesic, and the remaining edges are straight line segments; clearly Γ is convex, that is, the geodesic for any two points contained inside of Γ is also contained inside of Γ. The labelled distances are Euclidean distances. We have

$$\begin{aligned} m(\Gamma) &= \frac{c(\Gamma)}{l(\Gamma)} \\ &= \frac{\pi + 2\alpha + \frac{k}{h_1}}{\rho(x,u) + \rho(y,v) + \rho(u,v) + \rho(x,y)} \\ &= \frac{\pi + 2\arctan\frac{k}{2h_2} + \frac{k}{h_1}}{2\log\frac{h_2}{h_1} + \frac{k}{h_1} + \log\frac{|k+2ih_2|+|k|}{|k+2ih_2|-|k|}}. \end{aligned}$$

As $k \to \infty$, we see that $m(\Gamma) \to 1$. Let γ be the closed curve that begins at the vertex x, travels along the geodesic to y, and then travels along the same geodesic back to x. Then $m(\gamma) = \frac{2\pi}{2\rho(x,y)}$. As $k \to \infty$, $\rho(x,y) \to \infty$, and $m(\gamma) \to 0$. Therefore, there must be some k such that $m(\gamma) < m(\Gamma)$, and thus the DNA inequality is not true in the hyperbolic plane.

It is then natural to ask whether there is still some sort of analogue to the DNA inequality in the hyperbolic plane. We will look at the simplest case, the case where the cell Γ is the hyperbolic circle. The hyperbolic circle of radius r centered at the point c is defined as the set of points p such that $\rho(p,c) = r$. It is easy to show that the length of a hyperbolic circle of radius r is $l = 2\pi\sinh(r)$ and its area is $A = 2\pi(\cosh(r)-1)$. If Γ is a hyperbolic circle of radius r, then by symmetry its curvature κ is the same at all points. The Gaussian curvature K of $\mathbb{H}^2$ is -1, and the Euler characteristic of the circle is 1 [???]. The Gauss-Bonnet formula is

$$K \cdot A + l \cdot \kappa = 2\pi \cdot \chi$$

and therefore

$$\kappa = \coth(r),$$

the hyperbolic cotangent of r.

Since κ is constant, we conclude that $m(\Gamma) = \coth(r)$. Note the similarity here and in the spherical case.

Since $\coth(r)$ is a decreasing function on $(0,\infty)$, we arrive at the weak conclusion that the DNA inequality holds in the hyperbolic plane when both the DNA and the cell are circles. However, the inequality still does not hold in the hyperbolic circle in the hyperbolic plane in general. Indeed, suppose we let the cell Γ be a circle of radius r. Consider any point on Γ, and let γ be the curve that begins at this point, travels to the point $2r$ away on the opposite side of its diameter, and then travels back to the first point. Then we have $m(\gamma) = \frac{2\pi}{4r} = \frac{\pi}{2r}$. It is easy to verify numerically that for large r, we get $m(\gamma) > m(\Gamma)$.

6 Conclusion

We would like to thank Serge Tabachnikov and Misha Guysinsky of Pennsylvania State University for their guidance.

References

[1] I. Fáry. Sur certaines inégalités géométriques. *Acta. Sci. Math. Szeged* **12** (1950), 117-124.

[2] I. Fáry. Sur la courbure totale d'une courbe gauche faisant un nœud. *Bull. Soc. Math. France* **77** (1949), 128-138.

[3] S. Katok. Continued Fractions, Hyperbolic Geometry, and Quadratic Forms. *MASS Selecta*, American Mathematical Society, 2003.

[4] J. Lagarias, T. Richardson. Convexity and the Average Curvature of Plane Curves. *Geom. Dedicata* **67** (1997), 1-30.

[5] S. Tabachnikov. The Tale of a Geometric Inequality. *MASS Selecta*, American Mathematical Society, 2003.

[6] L. A. Santaló. *Integral Geometry and Geometric Probability*, Addison-Wesley, 1976.