BENFORD'S LAW AND RANDOM ITERATIONS OF $2x^2$ AND $3x^2$

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Abstract: I will outline the basics of ergodic theory to show that $O_T(x)$ is Benford for almost all x for random iterations of T(x)= $2x^2$ and T(x)= $3x^2$.

Introductory Ergodic Theory

(X,z,m) is a complete probability space with set X, s -algebra z, and measure m: m(C)=1, m is a countably additive non-negative function, and if A and B are disjoint m(AÜB)=m(A)+m(B). T is a transformation T:X®X, and T is measure-preserving, that is for every E in z, m(T^{-1} E)=m(E). The systems (X,z,m , T) are the fundamental objects of ergodic theory, but hereafter, I will refer only to transformations and the corresponding probability space will be implied.

T is **ergodic** if every invariant measurable function (f(T)=f a.e.) on X is constant a.e. that is, (X,z, m , T) is ergodic if and only if the orbit of almost every point visits each set of positive measure.

Lastly, the notion of **isomorphism** also occurs in ergodic theory. (X,z,m , T)≈(Y,b,n,S) means there exists f:C\ X_0 ®Y\ Y_0 where X_0 and Y_0 are merely sets of measure 0, fT=Sf on C\ X_0 and m(ϕ^{-1} E)=n(E) for every measurable E in Y\ Y_0 . Important to our discussion is that isomorphisms preserve ergodicity.

Finally, a theorem crucial in our discussion is the **Birkhoff Ergodic Theorem**.
It says for a measure-preserving map T, and integrable f, the following limit exists almost everywhere, and when it exists, the following equality holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(T^k x\right) = \int_X f(x)\, d\mu$$

Basically, it says that "time average" equals "space average."

Application of the Ergodic Theory

In our case, we analyze two probability spaces:

(X,m) where X=[0,1) and m is the Lebesgue measure and (Ω_2,m) where Ω_2 is the set of all sequences consisting of 0's and 1's and m is $\frac{1}{2^n}$ for n fixed members in a sequence. And the transformations of interest are H(x)=2x mod 1 over X and S the shifting transformation over Ω_2, that is if $\omega = \omega_0\omega_1\omega_2\omega_3\cdots$ then $S(\omega) = \omega_1\omega_2\omega_3\cdots$

Since our goal is to determine whether or not $O_T(x)$ is Benford, looking at the distributions of log($2x^2$) and log($3x^2$) mod 1 becomes the issue and this is merely log(x^2)+log(2) or log(x^2) +log(3) respectively, but log(x^2) is merely 2x on the circle so we can consider just 2x+log(2) and 2x+log(3).

I now wish to show that (X,m,H)≈(Ω_2,m,S).

Basically, for every x in X there corresponds exactly one w in Ω_2 when you consider the orbit of any x. That is, given that we know initially what half of the interval x is in, there are only two possible intervals for H(x) to fall; a 0 in the sequence corresponds to the left-most interval and a 1 in the sequence corresponds to the right; and if we have w'≠w then the correspond to different points since after enough iterations of H, x' and x will be in different intervals—specifically, if the nth member of w' differs from that of w $H^n(x')$ and $H^n(x)$ will be in different intervals. Likewise, if x'≠x then w'≠w because |x'-x|=e>0 so for some n $\frac{1}{2} \le 2^n \epsilon < 1$ (because there must exist some n such that $\frac{1}{2^{n-1}} \le \epsilon < \frac{1}{2^n}$) thus in the nth position, w' and w will have different entries because $H^n(x')$ and $H^n(x)$ will be in different intervals. Thus, (X,H)≈(Ω_2, S).

Now consider $X \times \Omega_2$ with the attached transformation T such that

$(x, \omega) \xrightarrow{T} (2x + \log A_{\omega_0} \bmod 1,\ S(\omega))$ where $A_{\omega_0} = \begin{cases} 2 \text{ if } \omega_0 = 0 \\ 3 \text{ if } \omega_0 = 1 \end{cases}$

This is equivalent to $(x, y) \xrightarrow{\overline{T}} (2x + \varphi(y) \bmod 1,\ 2y \bmod 1)$ where

$$\varphi(y) = \begin{cases} \log(2) \text{ if } y < \frac{1}{2} \\ \log(3) \text{ if } y \geq \frac{1}{2} \end{cases}$$ over X×X since $\Omega_2 \approx X$.

Through arguments similar to those above, it can be shown that $(X \times X, \bar{T}) \approx (\Omega_4, S)$ And since it is well-known that Ω_4 is ergodic and since isomorphisms preserve ergodicity, we have that $(X \times X, \bar{T})$ is ergodic. We can now apply Birkhoff’s theorem and we’ll choose f such that $f(x, y) = \chi_{[a,b]}(x)$ so we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[a,b]}\left(T^k x\right) = \int_X \chi_{[a,b]}\, dm = b - a$ for almost all x. And the ergodicity of $\bar{T}$ assures us that for almost every x, the orbit of x will visit every set of positive measure, so $O_{\bar{T}}(x)$ is uniformly distributed for almost every x, which is equivalent to $O_T(x)$ being Benford for almost all x for almost all orders of iterations of

$$T_i = \begin{cases} 2x^2 \\ 3x^2 \end{cases}$$.

Conclusion

Therefore, for random iterations of $2x^2$ and $3x^2$, we will almost always get a Benford sequence, and through similar argumentation, it should be easily shown that this may work in a much more general case where instead of 2 and 3 we have arbitrary constants, and perhaps even for more than just two possible mappings; and maybe even for similar mappings with a shift by a constant or even a multiple of x. Regardless of all this, however, we have shown with ergodic theory that the case at hand holds to Benford’s Law even though it does not satisfy the conditions necessary for it to follow from the theorem of Berger, Bunimovich, and Hill.