# GROUP STRUCTURE ON CANTOR RATIONALS

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ABSTRACT. In this article, we observe and establish the following interesting fact: the period length for the ternary expansion of a rational number divides the count of all Cantor rationals with the same denominator.

## 1. INTRODUCTION

The Cantor middle thirds set has been explored in some detail. Little work has been done, however, on the specific behavior of rational points therein. In order to prove our main theorem, we will construct a group which acts on the set of Cantor rationals of a given denominator. This group will divide our set into an integer number of partitions, each of which will be of size $n$ or $2n$, and our result will then follow as a corollary. Before we begin, though, we will need to equip ourselves with a few simple observations about the inherent symmetries on the Cantor rationals.

**Lemma 1.1.** *The following facts hold for rational points on the Cantor middle thirds set:*

(1) *A number is a rational point in the Cantor middle thirds set if and only if its ternary expansion is eventually periodic and contains no ones.*
(2) *The Cantor middle thirds set is invariant under reflection around the point $\frac{1}{2}$. We call the reflection, $R(\frac{p}{q}) = \frac{q-p}{q}$, the complement of $\frac{p}{q}$.*
(3) *The Cantor middle thirds set is invariant under division and multiplication by three, so long as the product still lies in the interval $[0, 1]$.*

*Proof.* A proof of the (1) is provided in [1], and (2) follows immediately from the symmetry of the set. (3) follows from the fact that multiplication and division by three simply translates the decimal point to the right or left, respectively, and thus preserves the no-ones condition. So long as multiplying by three does not land us outside our interval, it preserves our condition for Cantor rationality. $\square$

From these simple observations, we can already discern much about the structure of the rational points on the Cantor set with denominator $q$. For instance, it is evident from (3) that if $\frac{p}{q}$ is a Cantor rational, then $\frac{q-p}{q}$ is also. If we have $p \leq 3q$ then we also have that $\frac{3p}{q}$ is a Cantor rational. We will now introduce some definitions that will be helpful to us later.

**Definition 1.2.** Define $\mathbf{A_n}$ to be the set of Cantor rationals with denominator n. Define $|A_n|$ to be the number of elements in $A_n$.

Our next definition will be motivated by the fact that $A_q$ is nonempty if and only if $A_{3q}$ is also [4].

**Definition 1.3.** Define $A_q$ to be a **spawning set** for the Cantor rationals if $A_q$ is non-empty, and $q$ is not divisible by 3. Define the sets, $A_{3q}, A_{9q}, A_{27q}, A_{81q}, \dots$ to be the **child sets** of $A_q$.

Every nonempty $A_q$ is thus either a spawning set or is a child set for some previous spawning set.

## 2. n divides $|A_q|$

In this section we will introduce the group structure with which we prove our main result. In order to do this we first recall our transformation R which reflects points around $\frac{1}{2}$. Let $\breve{a}_i$ indicate the complementary digit to $a_i$. That is, if $a_i$ is a zero, then $\breve{a}_i$ is a two, and vice versa. Thus,

$$R(0.a_1a_2a_3\ldots) = 0.\breve{a}_1\breve{a}_2\breve{a}_3\ldots.$$

We will also need to introduce a new transformation, $T$, acting on the spawning sets which we will show to cyclically permute the digits in the period of its argument.

**Definition 2.1.** Assume $A_q$ is a spawning set, and let $\frac{p}{q} \in A_q$. Define the transformation $T : A_q \to A_q$ as follows:

$$T(\frac{p}{q}) = \begin{cases} 3\frac{p}{q}, & p \leq 3q \\ R(3R(\frac{p}{q})), & p > 3q. \end{cases}$$

*Remark* 2.2. Note that our lemma assures that $T$ maps each $A_q$ back onto itself. Since we assumed that $A_q$ was a spawning set, 3 does not divide q, and so we can be sure that $\frac{p}{q}$ has a purely periodic ternary expansion. Let's examine the effect of our transformation on this period. Let the elements of $A_q$ have period length $n$ (we know from number theory that these will all be the same). Then we may write,

$$\frac{p}{q} = 0.\overline{a_1\ldots a_n}.$$

The first case for the transformation is just equivalent to $a_1 = 0$. Multiplying by three, we shift all the digits once to the left, and it is evident that we have just cycled the digits in the period,

$$\frac{p}{q} = 0.\overline{0a_2\ldots a_n}$$
$$T(\frac{p}{q}) = 0.\overline{a_2\ldots a_n0}.$$

The second case is slightly more complicated. In this case, $a_1 = 2$, and so if we multiply by three, we will land outside our interval. So we play a trick. First we take the complement, yielding a zero for the first digit. We then multiply by three, and finally take the complement again:

$$T(\frac{p}{q}) = R(3R(\frac{p}{q})) \quad = \quad 0.\overline{a_2\ldots a_n2}.$$

and again, we have just cyclically permuted the digits. We will now present our primary theorem.

**Theorem 2.3.** *$T$ and $R$ generate a group action $G$ on each spawning set $A_q$ with the following properties: Let $n$ designate the period length of the elements of $A_q$. Then,*

1. *$T^n = I$, $R^2 = I$.*
2. *$T$ and $R$ commute.*
3. *$G$ is isomorphic to $Z_n \times Z_2$.*

(4) *$G_T$, the subgroup generated by $T$ alone, is a faithful cyclic subgroup of order $n$.*

*Proof.* (1) These are obvious.
(2) To see this, we again focus on the effects of $T$ and $R$ on the period of their argument. $T$ cycles the digits and $R$ swaps zeros and twos. Whether we cycle and then swap, or swap and then cycle, we get the same result.
(3) From (1) and (2) we may write any $g \in G$ as $T^i R^j$ for $i = 0, \ldots, n$ and $j = 0, 1$. Thus $\phi(T^i R^j) = (i, j)$ is the required isomorphism.
(4) Since the period of $\frac{p}{q}$ has no subperiod, we know that $T^m x = x$ implies m=n.

**Theorem 2.4.** *Each partition of $A_q$ by $G$ contains either $n$ or $2n$ elements.*

*Proof.* Since $G_T$ acts faithfully on the partition, the set $\{\frac{p}{q}, T(\frac{p}{q}), \ldots, T^{n-1}\}$ gives $n$ distinct elements, exhaustive of $G_T$. We then consider two cases.
Case 1. The entire group G acts faithfully. Then the size of the partition is $2n$, and we are done.
Case 2. G does not act faithfully on the partition. This statement is equivalent to the condition that $R(T^m)x = x$ for some $m < n$, $x \in A_q$. Then,

$$x = RT^m RT^m x = R^2 T^{2m} x = T^{2m} x.$$

Since the subgroup $G_T$ acts faithfully on the partition, this implies $m = n/2$. Recalling our original condition, we have that $RT^{n/2}(\frac{p}{q}) = \frac{p}{q}$. Thus, $\frac{p}{q}$ has period $a_1 \ldots a_n \breve{a}_1 \ldots \breve{a}_n$. In this case, it is plain that all the $R(T^m)$ are redundant, and so the partition is only of size $n$. □

*Remark* 2.5. Recall that we showed earlier that numbers with the above form belong to spawning sets $A_{3^n+1}$ for some integers $p$ and $n$.

**Corollary 2.6.** *Let $A_q$ be a spawning set whose elements have period length $n$. Then $n$ divides $|A_q|$.*

*Proof.* We have that $n$ divides each partition, and since there are an integer number of disjoint partitions in $A_q$, $n$ divides $|A_q|$.

**Corollary 2.7.** *$n$ divides $|A_q|$ for all $q$, not just for spawning sets.*

*Proof.* All the child sets have the same period length $n$. Their sizes are integer multiples of the spawning set size. Thus $n$ divides $|A_q|$ implies $n$ divides $|A_{3^k q}|$ $\forall$ $k \in \mathbb{Z}$. □

## References

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