RATIONAL POINTS IN THE CANTOR MIDDLE THIRDS SET

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ABSTRACT. In this paper we develop methods for determining the asymptotic behavior of rational points on the Cantor middle thirds set. Particularly we are concerned with how many reduced rational points with denominators from 1 to N lie on the set.

1. DEFINITIONS

To facilitate our exploration of rational points in the Cantor middle thirds set, we have included several key definitions.

Definition 1.1. A rational number $\frac{p}{q} \in \mathbb{Q}$ is a **Cantor rational** if it meets the following conditions:

(1) $\frac{p}{q}$ is in the Cantor middle thirds set, as defined in [1]
(2) p and q are relatively prime, i.e. $\frac{p}{q}$ is in reduced form.

Denote the Cantor rationals by the symbol $\mathbb{M}_Q$. Hereafter, abbreviate "$\frac{p}{q}$ is a Cantor rational" by $\frac{p}{q} \in \mathbb{M}_Q$.

Because we are working towards estimating the total number of Cantor rationals with denominator q, where $1 \leq q \leq N$, we must introduce the notation we will use to denote this sum.

Definition 1.2. Define $\mathbf{A_n}$ to be the set of Cantor rationals with denominator n. Define $|A_n|$ to be the number of elements in A_n.

Definition 1.3. Define $\mathbf{C_n}$ to be the total number of Cantor rationals with denominator q, where $1 \leq q \leq N$.

$$C_n = \sum_{i=1}^{N} |A_i|$$

The Cantor middle thirds set is constructed in a way that favors the use of base three over base ten. Therefore, we introduce the following definition.

Definition 1.4 ([1])**.** The sequence of integers $0.s_1 s_2 s_3 \ldots$ where each s_i is either $0, 1$, or 2 is called the **ternary expansion** of x if

$$x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$$

Representing numbers in this way provides us with a simple test for Cantor rationality. If a number's ternary expansion is eventually periodic, then that number is rational. If, further, that number has a ternary expansion containing no ones, then it is a Cantor rational. See [1] for more details.

Note that some values on $[0,1]$ have multiple ternary expansions. In cases where the period is $\overline{0}$ or $\overline{2}$ there is a potential ambiguity. For example,

$$\begin{aligned} 1 &= 1.000\ldots \\ &= 0.222\ldots \end{aligned}$$

In such cases, we choose the expansion containing no ones, if it exists. If both possible expansions of a number contain ones, then that number is not a Cantor rational. In the above example we may choose the second expansion, which contains no ones, and thus $1 \in \mathbb{M}_Q$.

In addition to using the ternary expansion of a number, we also found the 3-adic expansion to be a useful tool for determining Cantor rationality. It is important to note that using p-adic numbers was not necessary to our theorems (the reader may wish to develop the proofs for themselves using simpler considerations), but using p-adics made the proofs clearer and shorter. We will now briefly introduce the reader to the notion of p-adic numbers as described in [3].

Definition 1.5 ([3])**.** Let $p \in \mathbb{N}$ be any prime number. Define a map $|\cdot|_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} \frac{1}{p^{\mathrm{ord}_p x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where

$$\mathrm{ord}_p x = \begin{cases} \text{the highest power of p which divides x,} & \text{if } x \in \mathbb{Z} \\ \mathrm{ord}_p a - \mathrm{ord}_p b, & \text{if } x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0. \end{cases}$$

Call this map the p-adic norm.

Define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the p-adic norm. We call $\mathbb{Q}_p$ the **field of p-adic numbers**, where the elements of this field are equivalence classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm.

Specifically, we will work with 3-adic numbers.

Definition 1.6. The sequence of integers $d_{-m} \ldots d_{-1}.d_0 d_1 d_2 d_3 \ldots$, where each d_i is either 0,1, or 2 is called the **3-adic expansion** of a if

$$a = \sum_{i=-m}^{\infty} d_i p^i.$$

We know that the 3-adic expansion represents a rational number if and only if it is eventually periodic to the right [3].

There is no ambiguity in 3-adic expansions like that which we encountered with ternary expansions. It is a property of the p-adic metric that if two p-adic series converge to the same number, then they are the same series. Since the 3-adic expansion is just one such series, the 3-adic expansion of a number is unique [3]. We will frequently call on the fact that the ord_3 of a number is equivalent to the number of leading zeros in its 3-adic expansion.

In the terminology of the p-adic numbers, an expansion with no digits to the left

of the decimal represents a 3-adic integer. Many numbers which are not integers in $\mathbb{R}$ are integers in $\mathbb{Q}_p$. For instance,

$$\frac{1}{13} = .1\overline{220}$$

is an integer in the 3-adic field.

2. The 3-adic field and the Cantor rationals

With the following theorem, we intend to show that the test for Cantor rationality that we applied to ternary expansions can also be similarly applied to 3-adic expansions.

Theorem 2.1. *Let q be not divisible by three and $p \leq q$. Then $\frac{p}{q}$ is a Cantor rational if and only if the 3-adic expansion of $-\frac{p}{q}$ contains no ones.*

Proof. Since $-\frac{p}{q}$ is rational, with q not divisible by three, it has a purely periodic ternary expansion, $-\frac{p}{q} = -0.\overline{a_1 a_2 \ldots a_n}$, for some n. From elementary number theory, this means we may write

$$
\begin{aligned}
-\frac{p}{q} &= -\frac{a_1 3^{n-1} + a_2 3^{n-2} + \cdots + a_n}{3^n - 1} \\
&= \frac{1}{1 - 3^n}\left(a_n + \cdots + a_2 3^{n-2} + a_1 3^{n-1}\right).
\end{aligned}
$$

In the field of 3-adic numbers, we have that

$$\frac{1}{1 - 3^n} = \sum_{i=0}^{\infty} 3^{ni}.$$

Note that this sum is only convergent for 3-adic numbers. Thus,

$$
\begin{aligned}
-\frac{p}{q} &= \left(\sum_{i=0}^{\infty} 3^{ni}\right)\left(a_n + \cdots + a_2 3^{n-2} + a_1 3^{n-1}\right) \\
&= .\overline{a_n \ldots a_1},
\end{aligned}
$$

when expanded 3-adically. From this we see that the 3-adic expansion of $-\frac{p}{q}$ contains a one if and only if the ternary expansion of $\frac{p}{q}$ does, and we are done. $\square$

3. Some Helpful Observations

Lemma 3.1. *The following facts hold for rational points on the Cantor middle thirds set:*

(1) *The Cantor middle thirds set is invariant under reflection around the point $\frac{1}{2}$. We call the reflection, $R(\frac{p}{q}) = \frac{q-p}{q}$, the complement of $\frac{p}{q}$.*

(2) *The Cantor middle thirds set is invariant under division and multiplication by three, so long as the product still lies in the interval $[0,1]$.*

Proof. (1) follows immediately from the symmetry of the set. (2) follows from the fact that multiplication and division by three simply translates the decimal point to the right or left, respectively, and thus preserves the no-ones condition. So long as multiplying by three does not land us outside our interval, it preserves our condition for Cantor rationality. $\square$

Corollary 3.2. A_q *is nonempty if and only if* A_{3q} *is nonempty.*

4. Spawning Sets

We have described several general properties of Cantor rationals and have discussed the two ways that we will be representing them - 3-adically and in ternary. We continue with a definition that follows from the facts presented thus far, specifically from Corollary 3.2.

Definition 4.1. Define A_q to be a **spawning set** for the Cantor rationals if A_q is non-empty, and q is not divisible by 3. Define the sets, $A_{3q}, A_{9q}, A_{27q}, A_{81q}, \ldots$ to be the **child sets** of A_q.

After proving the relatively simple observations above, we ran several computer experiments and noticed that the elements in the spawning sets had a particular structure: exactly half of the elements had numerators divisible by three and these elements corresponded by reflection to those elements with numerators not divisible by three. The following theorem establishes that exactly half the elements of a spawning set are divisible by three.

Theorem 4.2. *Let* A_q *be a spawning set. Then* $\frac{p}{q} \in A_q$ *has p divisible by three if and only if its complement,* $\frac{q-p}{q}$, *has (q-p) not divisible by three.*

Proof. ($\Rightarrow$) Let p be divisible by three. Then using simple number theory, we know that if $3 \nmid q$ then $3 \nmid (q - p)$.
($\Leftarrow$) Conversely, let p be not divisible by three. Then certainly -p is also not divisible by three, and so it will have a non-zero number for its first 3-adic digit. Thus,

$$\mathrm{ord}_3(-\frac{p}{q}) = \mathrm{ord}_3(-p) - \mathrm{ord}_3(q) = \mathrm{ord}_3(-p) = \mathrm{ord}_3(p) = 0,$$

which implies that $\frac{p}{q}$ will also have a non-zero first 3-adic digit. Theorem 2.1 dictates, then, that this digit be a two. Adding one to both sides, it is clear that $R(\frac{p}{q}) = \frac{q-p}{q}$ has a zero for its first 3-adic digit, and thus,

$$\mathrm{ord}_3(q - p) = \mathrm{ord}_3(q - p) - \mathrm{ord}_3(q) = \mathrm{ord}_3(\frac{q - p}{q}) > 0,$$

as required.   □ Note that if A_q is a child set, then it contains no $\frac{p}{q}$ with p divisible by three, because it must have q divisible by three, and since all $\frac{p}{q} \in A_q$ must be in reduced form, p cannot be divisible by three.

We have thus far discussed some basic facts about rational points in the Cantor set and about rationals in the spawning sets. At this point,we are concerned with two questions: for which q is A_q a spawning set, and how many elements does each spawning set contain? We will start our exploration by examining for which general q A_q is a spawning set.

Theorem 4.3. $A_{\frac{3^n-1}{\phi}}$ *is a spawning set for all integers n, and all $\phi \in \mathbb{Z}$ such that:*

(1) $\quad \phi \mid 3^n - 1$

(2) $\quad \phi \mid \sum_{i=0}^{n-1} a_i 3^i$ *for some combination of $a_i = 0$ or 2.*

(3) $\quad \dfrac{\left(\sum_{i=0}^{n-1} a_i 3^i\right)/\phi}{(3^n-1)/\phi}$ *is reduced.*

Furthermore, all spawning sets are of this form for some ϕ.

Proof. Clearly $\frac{3^n-1}{\phi}$ is not divisible by three, and so it suffices to show that $A_{\frac{3^n-1}{\phi}}$ is nonempty. To show this, we need only consider the definition of the ternary expansion. If a $\frac{p}{q}$ has period n, this means that we may write

$$\frac{p}{q} \;=\; \frac{\sum_{i=0}^{n-1} a_i 3^i}{3^n - 1}.$$

where our condition for Cantor rationals is that each a_i be either a zero or a two. Since we assume that ϕ divides both the numerator and the denominator, for some combination of a_i, and we assume that after dividing both by ϕ, we have a reduced rational, we have that $A_{\frac{3^n-1}{\phi}}$ is nonempty, and thus it is a spawning set for all integers n and ϕ satisfying (1)-(3).

To show that all spawning sets are of this form, assume $\frac{p}{q}$ is an element of a spawning set. Then it has a purely periodic ternary expansion, and thus from elementary number theory we may write

$$\frac{p}{q} = \frac{\sum_{i=0}^{n-1} a_i 3^i}{3^n - 1},$$

where the a_i are the digits in the period of $\frac{p}{q}$.

Though we cannot identify all ϕ that will yield a spawning set in the above formula, we can definitively say that the following two formulas will be spawning sets for all integers n.

Corollary 4.4. $A_{\frac{3^n-1}{2}}$ *is a spawning set for all integers n.*

Proof. Note that $3^n - 1$ is always even, as are all the a_i. If we take the specific case, $a_0 = 2$, $a_i = 0 \; \forall \; i \neq 0$, $\phi = 2$, we get that $\frac{1}{3^n-1} \in \mathbb{M}_q$. Thus $A_{\frac{3^n-1}{2}}$ is a spawning set for all n.

Theorem 4.5. A_{3^n+1} *is a spawning set for all integers n.*

Proof. Certainly $3^n + 1$ is not divisible by three, and so it suffices to show that A_{3^n+1} contains at least one Cantor rational. We divert our attention momentarily

to ternary expansions where the period is of the following form: Let $\breve{a}_i = 2 - a_i$.

$$0.\overline{a_1 \breve{a}_1}$$
$$0.\overline{a_1 a_2 \breve{a}_1 \breve{a}_2}$$
$$0.\overline{a_1 a_2 a_3 \breve{a}_1 \breve{a}_2 \breve{a}_3}$$
$$\vdots$$
$$0.\overline{a_1 \ldots a_n \breve{a}_1 \ldots \breve{a}_n}$$

If we restrict each $a_i = 0$ or 2, then certainly sequences of this form represent Cantor rationals. We will see that these are precisely the expansions for $\frac{p}{3^n+1}$. Recalling the definition of the rational period, we may write the general case as follows:

$$
\begin{aligned}
0.\overline{a_1 \ldots a_n \breve{a}_1 \ldots \breve{a}_n} &= \frac{3^n (a_1 \ldots a_n) + 3^n - 1 - a_1 \ldots a_n}{3^{2n} - 1} \\
&= \frac{(3^n - 1)(a_1 \ldots a_n) + (3^n - 1)}{(3^n + 1)(3^n - 1)} \\
&= \frac{a_1 \ldots a_n + 1}{3^n + 1}.
\end{aligned}
$$

However, we cannot be sure that all combinations for the numerator will yield a rational in reduced form (in most cases, they won't). But since $a_i = 0 \ \forall \ i$ gives $\frac{1}{3^n+1}$, we know that A_{3^n+1} has at least one element, and thus A_{3^n+1} is a spawning set for all n. $\square$

Unfortunately we cannot answer in closed form the question of divisibility, and so we will only be able to predict the occurrence of spawning sets approximately. For the time being, then, let us turn our attention to the question of how many elements a given spawning set will contain.

5. RESULTS

Building on what we have established so far about spawning sets and their elements, we are ready to make a statement about how the number of Cantor rationals with denominators less than a given value will grow. Specifically we develop a strict minimum, which we can use to describe the asymptotic behavior of the Cantor rationals.

Theorem 5.1. *Let A_q be a spawning set. Then $|A_{3^k q}| = 2^{k-1}|A_q|$. for $k \geq 1$.*

Proof. Let A_q be a spawning set.
Case 1: Let $k = 1$. By Theorem 4.2, we know that precisely half the elements of A_q are divisible by three, and half are not. Thus, when we apply Theorem **??** to those that are divisible by three, they will correspond to other elements in A_q. And when we apply Theorem **??** to those elements of A_q that are not divisible by three, they will each correspond to a unique element in the child set, A_{3q}. By Theorem **??**, each of these elements has a unique complement, also in A_{3q}. Thus, $|A_{3q}| = 2 \cdot \frac{1}{2} \cdot |A_q| = |A_q|$.
Case 2: Let $k > 1$. By lemma **??** we know that none of the Cantor rationals in the child sets can be divisible by three. So, unlike in the case of spawning sets, Theorem

?? gives that *all* of the elements in $A_{3^k q}$ yield a unique element in $A_{3^{k+1} q}$. Again, by Theorem **??**, each of these new elements has a unique complement in $A_{3^{k+1} q}$. So each element in $A_{3^k q}$ corresponds to two elements in $A_{3^{k+1} q}$ by this construction, and so $|A_{3^{k+1} q}| = 2|A_{3^k q}|$. Combining cases 1 and 2, we have $|A_{3^k q}| = 2^{k-1}|A_q|$. $\square$

From these two theorems, we can calculate precisely a very useful quantity. Let A_i be a spawning set, and take Ω_i^N to be the sum,

$$\Omega_i^N = |A_i| + |A_{3i}| + |A_{9i}| + \ldots + |A_{3^{\lfloor \log_3(\frac{N}{i})\rfloor} i}|.$$

This sums the contributions of the spawning set A_i and all its child sets with denominators less than N. We can simplify Ω_i^N rather nicely.

$$
\begin{aligned}
\Omega_i^N &= |A_i| + |A_{3i}| + |A_{9i}| + \ldots + |A_{3^{\lfloor \log_3(\frac{N}{i})\rfloor} i}| \\
&= |A_i| + |A_i| + 2|A_i| + \ldots + 2^{\lfloor \log_3(\frac{N}{i})\rfloor - 1}|A_i| \\
&= |A_i| \left(1 + \sum_{k=0}^{\lfloor \log_3(\frac{N}{i})\rfloor - 1} 2^k \right) \\
&= |A_i| 2^{\lfloor \log_3(\frac{N}{i})\rfloor}
\end{aligned}
$$

If we observe that $\lfloor x \rfloor \geq x - 1$, we get a very nice formula for Ω_i^N.

$$
\begin{aligned}
\Omega_i^n &= |A_i| 2^{\lfloor \log_3(\frac{N}{i})\rfloor} \\
&\geq |A_i| 2^{\log_3(\frac{N}{i}) - 1} \\
&= \frac{|A_i|}{2} \exp(\log(\frac{N}{i})\frac{\log 2}{\log 3}) \\
&= \frac{1}{2} N^{\frac{\log 2}{\log 3}} |A_i| i^{-\frac{\log 2}{\log 3}}.
\end{aligned}
$$

Then, our original quantity, C_N is simply the sum of all the Ω's,

$$C_N \geq \frac{1}{2} N^{\frac{\log 2}{\log 3}} \sum_{i=1}^{N} |S_i| i^{-\frac{\log 2}{\log 3}},$$

where here the S_i are spawning sets. The next task will be to discover the asymptotic behavior of this quantity.

REFERENCES

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