MATRICES WITH NON-CONJUGATE CENTRALIZERS

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Let G be a group, and g an element in G. The set $Z_G(g) = \{h \in G \mid hg = gh\}$ is called the *centralizer* of g in G. (It is an easy exercise to prove that $Z(g)$ is a subgroup of G.)

Two subgroups G_1 and G_2 in G are called *conjugate* in G if there exists an element $g \in G$ such that $G_2 = g^{-1}G_1g$.

Project description. *Let $G = GL(n, \mathbb{Z})$. Find two matrices A and B in $SL(n, \mathbb{Z}) \subset G$ ($n \geq 3$) with the same characteristic polynomial such that $Z_G(A)$ and $Z_G(B)$ are not conjugate in G.*

Even for $n = 2$ the question is not trivial. The characteristic polynomial of a matrix $A \in SL(2, \mathbb{Z})$ has the form $x^2 - \mathrm{tr}(A)x + 1$, where $\mathrm{tr}(A)$ is the trace of A, and we will be interested in the case when A is *hyperbolic*, i.e. $\mathrm{tr}(A)^2 - 4 > 0$. If the matrix A is *primitive*, i.e. not a power of another matrix in $SL(2, \mathbb{Z})$, $Z_{SL(2,\mathbb{Z})}(A) = \{\pm 1\} \times \{A^n \mid n \in \mathbb{Z}\}$. The approach to this problem is based on a bijection between conjugacy classes of hyperbolic matrices in $SL(2, \mathbb{Z})$ of a given trace and ideal classes in the corresponding real quadratic field. The underlying algebraic concept is the notion of *class number*, and the problem is solved by a so–called *Gauss Reduction Algorithm* [1]. The answer depends on the trace: e.g. all matrices in $SL(2, \mathbb{Z})$ with trace 3 are conjugate already in $SL(2, \mathbb{Z})$, and for the trace 4, the matrices $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ are not conjugate in $SL(2, \mathbb{Z})$, but conjugate in $GL(2, \mathbb{Z})$. The problem for $n \geq 3$ can be also translated into the language of algebraic number theory since the above bijection persists in this case. It has been proved in [2] that the required example exists for some hyperbolic matrices, in particular, for the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 10 & 9 & 0 \end{pmatrix}$. The task is to develop an algorithm for finding a matrix B with centralizer not conjugate to that of A.

REFERENCES

[1] S. Katok, *Coding of closed geodesics after Gauss and Morse*, Geometriae Dedicata **63** (1996), 123-145

[2] A. Katok and S. Katok, *Rigidity of measurable structure for algebraic actions of higher–rank abelian groups*, preprint `http://www.math.psu.edu/katok_s/pub/actions.ps`