

PROBLEMS REU 2014

The problems are selected from several topics: convexity in extremal algebras, tilings and dynamical systems. Smaller groups of students (two or three) will work on particular problems, and larger clusters of students will work on a particular topic.

Problems related to convexity in extremal algebras

Problem 1. We will introduce max-plus segments and max-plus convex sets. Recall that a max-plus hemispace is a max-plus convex set with max-plus complementary. Generating sets for max-plus hemispaces are described in 2). The problem asks to classify finite families of basic max-plus convex sets, that is, finite families of convex sets for which the union is the whole max-plus algebra. Some progress was done in 3), research done during PSU REU 2013, where it is shown that hemispaces are polyhedral sets, with some faces of the boundary missing. A similar description is expected for a general basic set. This problem is an introduction to tropical mathematics, a topic of current research interest. See 4).

Related bibliography:

- 1) V. Nitica, I. Singer, Max-plus convex sets and max-plus semispaces I, Optimization 56 (2007) 171–205
- 2) R. Katz, V. Nitica, S. Sergeev, Characterization of tropical hemispaces by (P, R) -decompositions, Lin Alg Appl 440 (2014) 131–163
- 3) D. Ehrmann, Z. Higgins, V. Nitica, The geometric structure of max-plus hemispaces, arXiv:1402.2857
- 4) X. Allamigeon, S. Gaubert, E. Goubault, Computing the extreme points of tropical polyhedra, E-print arXiv:math/0904.3436v2

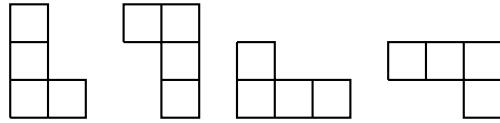
Problem 2. We will introduce max-min segments and max-min convex sets. Recall that a max-min hemispace is a max-min convex set with max-min complementary. The problem asks to find the geometric structure of max-min hemispaces. Related results for max-plus hemispaces are found in 3) and 4). A difficulty here is the fact that separation of convex sets by hyperplanes is not valid in max-min convexity. Some progress was done during REU 2013, when the structure of a closed max-min hemispace was studied. Nevertheless, the general case is still open.

Related bibliography:

- 1) V. Nitica, I. Singer, Contributions to max-min convex geometry I. Segments, Linear Algebra and its Applications 428 (2008) 1439–1459
- 2) V. Nitica, I. Singer, Contributions to max-min convex geometry II. Semispaces and convex sets, Linear Algebra and its Applications 428 (2008) 2085–2115
- 3) R. Katz, V. Nitica, S. Sergeev, Characterization of tropical hemispaces by (P, R) -decompositions, Lin Alg Appl 440 (2014) 131–163
- 4) D. Ehrmann, Z. Higgins, V. Nitica, The geometric structure of max-plus hemispaces, arXiv:1402.2857

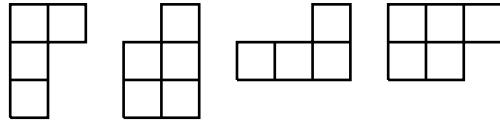
Tiling problems

Problem 3. In this problem, only translations of the tiles are allowed. Consider the family L_4 of tiles:



It was shown in [1] (paper written by Penn State REU 2012 participants) that a rectangle of even sides, and more general the first quadrant, can be tiled by L_4 only if the tiling reduces to a tiling by 2×4 and 4×2 rectangles. More general, a similar property was shown in [2] to hold for the family L_n (ribbon L -shaped polyominoes built out of n cells) if n is even. Another related paper is [3], paper written by Penn State REU 2013 participants.

The problem asks to investigate similar rigidity phenomena for other families of tiles. In particular, we believe that the set of tiles shown below can lead to undecidable tiling problems. For example, it is true that the plane can be tiled by the set below only if the tiling reduces to a tiling of the plane by 3×3 rectangles?



The group working on this problem will try to show for half of the period that the problem can be solved, and for the rest of the period, will try to show that the problem is undecidable.

Related bibliography:

- 1) M. Chao, D. Levenstein, V. Nitica, R. Sharp, A coloring invariant for ribbon L -tetrominoes, Discrete Mathematics 313 2013 611-621
- 2) V. Nitica, A rigidity property of ribbon L -shaped n -ominoes, submitted to Electronic Journal of Combinatorics
- 3) A. Calderon, S. Fairchild, M. Muir, V. Nitica, S. Simon, Rigid tilings of quadrants by L -ominoes and notched rectangles, submitted to Contributions to Discrete Mathematics
- 4) polysolver, web resource
<http://www.jaapsch.net/puzzles/polysolver.htm>

Dynamical systems problems

Problem 4. Let $f : M \rightarrow M$ be a Hölder topologically transitive Anosov diffeomorphism or shift of finite type. Consider C^r extensions that have as fiber certain semidirect products of compact and nilpotent Lie groups. Prove that those that are topologically transitive are open and dense.

Some partial results can be found in 3). The fiber in that case is a perfect Lie group for which there exists a dense and open set of families of generators. In the remaining cases, this condition does not hold. It is possible that certain multiple recurrence results may be applied here.

The problem is part of a general conjecture that is described in 4).

Related bibliography:

- 1) A. Katok, V. Nitica, Differential rigidity of abelian group actions. I, Introduction to cocycle problem, Cambridge University Press 2011
- 2) V. Nitica, M. Pollicott, Transitivity of Euclidean extensions of Anosov diffeomorphisms, Ergod. Th. & Dynam. Sys. 25 (2005), 257–269
- 3) V. Nitica, Stably transitivity for extensions of hyperbolic systems by semidirect products of compact and nilpotent Lie groups, Discrete and Continuous Dynamical Systems 29 (2011) 1197–1203
- 4) V. Nitica, I. Melbourne, A. Török, Stable transitivity of certain noncompact extensions of hyperbolic systems, Annales Henri Poincaré 6 (2005), 725–746

Problem 5. There are by now many genericity results about (stable) topological transitivity of extensions of hyperbolic dynamical systems by Lie groups. The proposed problem is to investigate the possibility of prevalency results in the sense introduced in 5).

Related bibliography:

- 1) A. Katok, V. Nitica, Differential rigidity of abelian group actions. I, Introduction to cocycle problem, Cambridge University Press 2011
- 2) V. Nitica, M. Pollicott, Transitivity of Euclidean extensions of Anosov diffeomorphisms, Ergod. Th. & Dynam. Sys. 25 (2005), 257–269
- 3) V. Nitica, Stably transitivity for extensions of hyperbolic systems by semidirect products of compact and nilpotent Lie groups, Discrete and Continuous Dynamical Systems 29 (2011) 1197–1203
- 4) V. Nitica, I. Melbourne, A. Török, Stable transitivity of certain noncompact extensions of hyperbolic systems, Annales Henri Poincaré 6 (2005), 725–746
- 5) W. Ott, J.A. Yorke, Prevalence, Bull. AMS 42, (2005) 263–290

Problem 6. Consider $S \subset \mathbb{R}$ containing both positive and negative values. Then the closure of the semigroup generated by S is a group. An analog of this property was shown to hold in other Lie groups: compact, linear, nilpotent, solvable and semidirect products of the above. For example, see 3), paper written by Penn State REU 2013 participants. We would like to investigate this so called *semigroup problem* in certain semisimple Lie groups

G of low dimension such as $SL(2, \mathbb{R})$. That is, we want to find conditions for $S \subset G$ such that the closure of the semigroup generated by S is a group.

Related bibliography:

- 1) V. Nitica, I. Melbourne, A. Török, Stable transitivity of certain noncompact extensions of hyperbolic systems, *Annales Henri Poincaré* 6 (2005), 725–746
- 2) V. Nitica, M. Pollicott, Transitivity of Euclidean extensions of Anosov diffeomorphisms, *Ergod. Th. & Dynam. Sys.* 25 (2005), 257–269
- 3) K. Lui, V. Nitica, S. Venkatesh, The semigroup problem for central semidirect product of R^n with R^m , *Topology Proceedings* 45 (2015) 9–29